Estimation and Inference in a Peer Effects Model under Heteroskedasticity

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Abstract

This paper develops estimation and inference for a panel-data peer-effects model with an unobserved individual-specific characteristic and heteroskedastic errors. The non-linear least squares (NLLS) estimator widely used in this literature is inconsistent under heteroskedasticity, with bias of indeterminate sign. We propose a cross-fit correction that delivers a consistent estimator robust to heteroskedasticity, and we provide the first analytic standard error for the non-linear peer-effects estimator, while the literature has relied on wild-bootstrap procedures. We apply the method to two empirical settings. In university transcript data from the COVID-19 online semester, NLLS finds a positive and significant classroom peer effect, whereas our estimator finds it close to zero and statistically insignificant; in the Italian matched employer–employee panel, by contrast, our estimate is about 15% larger than the NLLS estimate. The bias-corrected wage-variance decomposition further reveals that average coworker quality explains a share of wage variance comparable to that of firm effects — a channel the standard AKM model cannot see — with worker–coworker sorting as the dominant margin.

KEYWORDS: Social interactions, Spillover effects, Panel data, Non-linear regression, Cross-fitting, Leave-out estimation, Heteroskedasticity.
JEL CODES: C18, C23, I21, J31.

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1 Introduction

Many empirical studies of social interactions document strong correlations between individuals' outcomes and those of their peers. As Manski (1993) emphasizes, three distinct mechanisms can generate such patterns: contextual spillovers from exogenous peer characteristics, endogenous effects from peer outcomes, and correlated effects from shared characteristics or environments (De Paula, 2017). Contextual and endogenous spillovers cannot in general be separated when interactions take place within groups — the reflection problem of Manski (1993) — while identification is restored by variation in group sizes (Lee, 2007; Graham, 2008) or by structuring interactions through a network (Bramoullé, Djebbari and Fortin, 2009). Panel data extend this logic by allowing correlated effects to operate through an unobserved individual-specific characteristic, and Mas and Moretti (2009) and Arcidiacono, Foster, Goodpaster and Kinsler (2012) independently develop estimation procedures for this setting, the non-linear least squares (NLLS) estimator of the latter becoming the workhorse of a large applied literature.[1]

Peer and spillover effects of this kind have been documented across many domains — from classroom achievement and workplace productivity to urban agglomeration, regional labor markets, and firm-to-firm spillovers (e.g., Mas and Moretti, 2009; Cornelissen, Dustmann and Schönberg, 2017; Battisti, 2017; Burke and Sass, 2013; Thiemann, 2022; De la Roca and Puga, 2017; Dix-Carneiro and Kovak, 2017; Hong and Lattanzio, 2025; Portugal, Reis, Guimarães and Cardoso, 2024; Holden, Keane and Lilley, 2021; Messina, Sanz-de Galdeano and Terskaya, 2026; Neururer and Sun, 2021). In panel settings, the NLLS estimator of Arcidiacono et al. (2012) has become the standard tool for this design. Its consistency, however, rests on an assumption of homoskedasticity that is untenable in the administrative-data settings where it is typically applied, in which error variances differ across individuals through mobility, selection, and measurement heterogeneity. Inference, likewise, has relied on wild-bootstrap procedures, with no analytical alternative available for this class of estimators.

This paper develops estimation and inference for the panel-data peer-effects model when the errors are heteroskedastic. We establish three main theoretical results. First,

[1] In linear-in-means models, consistent estimation can also be facilitated by maximizing a Gaussian likelihood (Lee, Liu and Lin, 2010) or by a two-step instrumental-variables estimator (Kelejian and Prucha, 1998; Lee, 2003).

the NLLS estimator widely used in this literature is inconsistent under heteroskedasticity, and the direction of the bias is determined entirely by the empirical design — meaning practitioners cannot use NLLS as an upper or lower bound on the peer coefficient without correction. Second, we propose a cross-fit (CF) correction to the NLLS moment that yields a consistent estimator under heteroskedasticity. Third, we propose an analytic standard error for the peer coefficient, while the literature has relied on wild-bootstrap procedures for this class of estimators.

We first revisit the classroom peer-effect setting of Arcidiacono et al. (2012) using the universal transcript data from the University of Wisconsin–Madison. In a pre-pandemic benchmark sample, the CF estimate is about 30% smaller than the NLLS estimate (0.17 versus 0.25), and Monte Carlo simulations calibrated to this design show why the correction matters for inference as well: the nominal 95% confidence interval covers the truth in only 57% of draws under NLLS with the wild bootstrap, against 93% under our cross-fit estimator with the proposed standard error. As a further test, we turn to the Spring 2020 semester, when the pandemic moved all undergraduate teaching online. NLLS continues to return a positive and statistically significant classroom peer effect, whereas the CF estimate is close to zero and statistically insignificant.

We then apply our estimator to study workplace peer effects by incorporating a coworker component into a canonical AKM model (Abowd, Kramarz and Margolis, 1999a) using the Italian matched employer–employee panel. The cross-fit estimate is about 15% *larger* than the iterated NLLS estimate on the same sample — the opposite direction from the classroom application. With a consistent estimate of the peer coefficient in hand, we extend the canonical two-way worker–firm wage decomposition to incorporate average coworker quality. Two findings emerge. First, coworker quality explains about 11% of wage variance — a share comparable to that explained by firm effects, but invisible to the two-way model. Second, the worker–firm sorting correlation falls from 0.21 to 0.08 once coworkers are included, while a substantially larger worker–coworker correlation of 0.60 emerges in its place.

This paper contributes to three strands of the literature. The first is estimation *and* inference for peer effects operating through peers' unobserved heterogeneity. On estimation, we propose a cross-fit correction that yields a consistent estimator under general heteroskedasticity, whereas the NLLS estimator of Arcidiacono et al. (2012) is consistent only under homoskedasticity. Two related estimators take different routes.

The two-step estimator of Mas and Moretti (2009) addresses similar problems in long panels but is not consistent in the short-panel setting on which we focus, and Braun and Verdier (2023) proposes an instrumental-variables estimator, whereas ours requires no search for instruments. On inference, we provide the first analytic standard error for a non-linear peer-effects estimator, while the literature has relied on the wild bootstrap.

The second is a methodological literature on estimation and inference in models with many parameters (see Anatolyev, 2019, for a survey). This work has focused on linear specifications, building on leave-one-out constructions (Hausman, Newey, Woutersen, Chao and Swanson, 2012; Kline, Saggio and Sølvsten, 2020a; Anatolyev and Sølvsten, 2020) that purge incidental-parameter bias from second moments of estimated fixed effects. Our cross-fit correction adapts these ideas to a setting where the parameter of interest enters non-linearly, through an estimated peer average that is itself a function of the individual effects. More broadly, the correction applies to non-linear least squares problems in which a low-dimensional parameter multiplies a latent individual effect; models of heterogeneous teacher value-added (Hahn, Singleton and Yildiz, 2023; Kinsler, 2016) share this structure, so the method is not specific to peer averaging.

Finally, our workplace application contributes to a large literature on the sources of wage inequality, which has emphasized firm heterogeneity and worker–firm sorting as central drivers (Card, Heining and Kline, 2013; Card, Cardoso, Heining and Kline, 2018; Song, Price, Guvenen, Bloom and Von Wachter, 2019). Incorporating coworker quality into this decomposition has not previously been feasible: it requires a consistent peer coefficient together with second moments of the estimated coworker quality that are purged of incidental-parameter bias — the coworker-channel analog of the limited-mobility bias correction in two-way models (Andrews, Gill, Schank and Upward, 2008; Kline et al., 2020a). Earlier methods provide neither. Our correction supplies both, in effect opening the firm "black box" to separate the role of coworkers from that of the firm itself. We find that coworker quality rivals firm effects in its contribution to wage variance and that the dominant sorting margin shifts from worker–firm to worker–coworker — a reallocation that reshapes the standard account of how labor markets generate wage inequality. The classroom application, in turn, provides a heteroskedasticity-robust replication of the canonical Arcidiacono et al. (2012) setting that yields a meaningfully smaller estimated peer effect.

The paper is organized as follows. Section 2 sets out the model and the formal identification result. Section 3 discusses the source of inconsistency in NLLS under heteroskedasticity and introduces our cross-fit estimator with its consistency theorem. Section 4 presents the consistent variance estimator for valid inference. Section 5 reports the UW transcript application and Monte Carlo simulations calibrated to it. Section 6 reports the Italian workplace application and the extended wage-variance decomposition. Section 7 concludes. Implementation details, derivations and proofs, the endogenous-effects empirical extension, and a data appendix for the workplace application are relegated to the Appendix.

2 Model and identification

The primary theoretical contribution of the paper is to propose a novel cross-fit correction for the least squares estimator in a non-linear regression model where the number of regressors may be large. While the proposed approach applies broadly to such settings as described in Section 3, the current section introduces a motivating example which is the estimation of peer effects in a panel data model for wages. We return to this example in the empirical application of Section 5.

2.1 Contextual peer effects in unobservables

Consider the following framework. The outcome variable y_{it} denotes observed log wage for an individual i at time t. We are interested in the relationship between y_{it} and the average quality of individual i's contemporaneous group of peers. The peer group is observed by the researcher and is denoted by the index set $\mathcal{P}_{it} \subseteq \{1, \ldots, N\}$. The quality of each peer is unobserved but assumed to be captured by a measure of permanent ability α_i that also affects wages directly. Due to the possibility of endogenous sorting into peer groups, it is necessary to control for a vector of observed covariates w_{it}. As is common in applied practice, w_{it} may include a collection of group indicators. With additive separability, these considerations lead to a non-linear panel data regression

$$y_{it} = \alpha_i + \bar{\alpha}_{(i)t} \cdot \beta_0 + w_{it}'\gamma + \varepsilon_{it}, \qquad i = 1, \ldots, N,\ t = 1, \ldots, T_i, \tag{1}$$

where $\bar{\alpha}_{(i)t} = |\mathcal{P}_{it}|^{-1} \sum_{\iota \in \mathcal{P}_{it}} \alpha_\iota$ is the average of individual effects among i's peers. The object of interest is the coefficient on the average quality of the peers, $\beta_0 \in (-1, 1)$, while γ and $\alpha = (\alpha_1, \ldots, \alpha_n)'$ are non-random vectors of nuisance parameters.

Frameworks of the kind given above are widely used to determine the importance of peers in educational performance (e.g., Jackson and Bruegmann, 2009; Arcidiacono et al., 2012), wage settings (e.g., Lengermann, 2002; Cornelissen et al., 2017; Hong and Lattanzio, 2025), worker's productivity (e.g., Mas and Moretti, 2009; Guryan, Kroft and Notowidigdo, 2009; Brune, Chyn and Kerwin, 2020), firm revenues (e.g., Baum-Snow, Gendron-Carrier and Pavan, 2024).

The control variables w_{it} are included in the regression to ensure that no relevant confounders are excluded from the model so that the strict exogeneity of the peer group is satisfied. Letting $\mathcal{F}_i = \{w_{it}, \mathcal{P}_{it}\}_{t=1}^{T_i}$ collect individual i's observed history of peer groups and control variables, strict exogeneity can be formulated as

$$\mathbb{E}[\varepsilon_{it} \,|\, \mathcal{F}_i] = 0, \qquad\qquad i = 1, \ldots, N, \; t = 1, \ldots, T_i. \tag{2}$$

The set of control variables needed to ensure that (2) is satisfied depends on the specific context. We therefore have few further general comments about the choice of w_{it}.[2] Specifically, we consider the following specification for the controls.

$$w_{it}'\gamma = \psi_{j(i,t)} + \lambda_t + c_{it}'\gamma_c, \tag{3}$$

where $\psi_{j(i,t)}$ is the location effect (e.g., classroom or firm), λ_t is the time effect, and c_{it} is the observed time-varying individual characteristics. The inclusion of $\psi_{j(i,t)}$ controls for endogenous selection into firms or classrooms as in the seminal specification introduced by Abowd, Kramarz and Margolis (1999b), which is originally used in wage regression but has been widely adopted in many other settings.

In current empirical practice, estimation of (1) is commonly carried out by the use of (non-linear) least squares. Consistency of the resulting estimator for β_0 was established by Arcidiacono et al. (2012) in a setting with no control variables and

[2] When peers are completely randomly assigned, there is typically no need for any control variables. However, in many cases, random assignment is done conditional on a set of observed characteristics (e.g., Guryan et al., 2009), in which case it is most often necessary to include those characteristics in the model. With observational data, a judicious choice of control variables is typically required. Still, when interactions occur in groups, it may often be sensible to include a group fixed effect in $w_{it}'\gamma$ to further address the issue of correlated effects.

serially uncorrelated, homoscedastic error terms. To shed light on the role played by these assumptions, Section 3 discuss why consistency of least squares fails when the error terms are not serially uncorrelated and homoscedastic. Given that the error covariance structure is rarely so well-behaved, we propose a cross-fit correction to the least squares estimator that allows for some dependence and unrestricted heteroscedasticity. An implication of our negative result regarding least squares and the structure of our proposed estimator is that researchers need to be explicit about their assumptions on the error variance structure when considering their choice of point estimator.

The regression structure of (1) makes the interpretation of β_0's magnitude and sign canonical: β_0 captures a return to peer quality in the sense that a one unit increase in average peer quality corresponds to a $\beta_0 \cdot 100\%$ increase in wages (on average). However, as peer quality is unobserved, the meaning of a one unit increase is ambiguous, so it is important to supplement any estimate of β_0 with a summary of possible changes in peer quality. Towards this end, we adapt the proposal in Kline et al. (2020a) to provide an estimator of the overall variance in average peer group quality. We thereby facilitate that our proposed estimator of β_0 can be related to a standard deviation increase in average peer quality – a common way of grounding the interpretation of magnitudes in applied research. In practice, the mechanism through which peers affect outcomes is as interesting as the magnitude of the effect. Plausible mechanisms include knowledge spillover (Nix, 2016), peer pressure (Mas and Moretti, 2009), and promotion competition (Bianchi, Bovini, Li, Paradisi and Powell, 2021). The focus of this paper is the statistical problems of estimation and inference, so we will not delve further into the specific mechanisms that may drive the magnitude and sign of β_0.

2.2 Identifying variation

Identification of β_0 requires variation in average peer-group quality that cannot be predicted by the linear part of (1). Two conditions are needed. First, some individuals must have a time-varying peer group — that is, there must be *mobility* between observed peer groups; this condition is directly verifiable from the data by inspecting the share of movers and stayers and the rate of peer-group turnover. Second, mobility-induced compositional changes must actually move the unobserved average

peer quality $\bar{\alpha}_{(i)t} = |\mathcal{P}_{it}|^{-1} \sum_{\iota \in \mathcal{P}_{it}} \alpha_\iota$. The second condition cannot be verified ex ante because α is unobserved, but it is implied by any setting with non-degenerate individual heterogeneity and labor-market frictions that prevent perfect sorting (e.g., Mortensen and Pissarides, 1994; Postel–Vinay and Robin, 2002). In the extreme case of homogeneous α, identification is bound to fail; in any setting with some individual heterogeneity, observed mobility induces identifying variation in the unobserved average peer quality. A simple example with three individuals and two firms makes these conditions concrete and provides a useful entry point for understanding the cross-fit estimator we propose in Section 3: in this special case, our estimator reduces to a clean instrumental-variables estimator that uses two independent noisy measurements of the latent peer-quality regressor as mutual instruments, a familiar measurement-error correction. The full development is in Appendix B.1, and the connection to the general construction is made explicit in Section 3.

3 Estimation

This section starts by characterizing the source of inconsistency in the least squares estimator when applied to a generic regression model with multiplicative non-linearity. Described at a high level, the source of inconsistency is that the least squares objective function is not minimized near the truth, or equivalently, that the gradient of the objective does not have a zero near β_0. Using this observation as a starting point, the section then proposes a new estimator that sets a recentered gradient of the least squares objective function equal to zero.[3]

We now suppress the multiple subscripts that were used to facilitate an economic discussion of the peer effects example introduced in Section 2. We therefore consider a regression model with multiplicative non-linearity of the form

$$y_\ell = x_\ell'\delta + a_\ell'\delta \cdot \beta_0 + \varepsilon_\ell, \qquad \ell = 1, \ldots, n. \tag{4}$$

Here x_ℓ and a_ℓ are observed K-dimensional vectors, δ is a vector of nuisance parameters, and β_0 remains the object of interest.

In a peer effects setting, δ contains fixed effects and coefficients on control vari-

[3] As discussed further below, consistency also requires that there is sufficient identifying variation in average peer quality.

ables, x_ℓ contains dummy variables for fixed effects and control variables, and a_ℓ is a function of the peer group that observation ℓ belongs to and is thus dependent across ℓ. To encompass this example, we therefore do not impose restrictions on the dependence in x_ℓ and a_ℓ across ℓ. Instead, we conduct the analysis conditional on the regressors, $A = (a_1, \ldots, a_n)'$ and $X = (x_1, \ldots, x_n)'$, so that a_ℓ (and x_ℓ) may be arbitrarily dependent across observations.

The primary maintained assumptions are strict exogeneity, compactness of the parameter space for β_0, and a collection of full-rank conditions.

Assumption 1 (Primary conditions)**.** *(i)* Strict exogeneity: $\mathbb{E}[\varepsilon_\ell \mid X, A] = 0$ *for all* ℓ*, and* $\text{range}(X)$ *contains the constant vectors.*

(ii) Compact parameter space: $\beta_0 \in$ *interior*$(\mathcal{B})$*, where* $\mathcal{B} \subseteq \mathbb{R}$ *is compact.*

(iii) Full rank: *there exists* $N < \infty$ *such that* $(X + A\beta, A\delta)$ *has full rank for any* $\beta \in \mathcal{B}$ *and all* $n \geq N$*.*

(iv) Bounded fourth moments: *there exist constants* $C_4 < \infty$ *and* $N < \infty$ *such that, for all* $n \geq N$*,*

$$\max_{\ell \leq n} \mathbb{E}[\varepsilon_\ell^4 \,|\, X, A] \leq C_4.$$

Part (i) is a strict exogeneity condition, part (ii) restricts the true β_0 to be in the interior of a compact set $\mathcal{B}$ as is standard for non-linear models, part (iii) is a collection of full-rank conditions on implied matrices of regressors, and part (iv) is a standard uniform boundedness condition on the fourth moments of the error terms. Part (i) is often called "exogenous mobility" in the employer-employee matched data literature; it excludes sorting between individuals and peer groups based on latent effects that are not captured by fixed effects controlled for in the regression. Part (iii) encapsulates two restrictions on the design. The first restriction excludes multicollinearity among the entries in $x_\ell + a_\ell\beta$ for any β in the parameter space, and this condition ensures invertibility of the design matrix for estimating δ when β_0 is equal to β: $S(\beta) = \sum_{\ell=1}^n (x_\ell + a_\ell\beta)(x_\ell + a_\ell\beta)'$. The second restriction is that the "unobserved regressor" $a_\ell'\delta$ contains identifying variation, i.e., that $a_\ell'\delta$ varies in ways that are not fully captured by a linear combination of $x_\ell + a_\ell\beta$ for any $\beta \in \mathcal{B}$. In the context of peer effects models, this part of Assumption 1 was discussed in Section 2.2 and requires that the sample contains variation in peer group quality that is not completely explained by the control variables.

Remark 1. We note that the model (4) also accommodates the specification studied in Hahn et al. (2023), where a parameter of interest is β_0 capturing the extent of heterogeneity in teacher value-added:

$$y_{js} = w_{js}\gamma + \alpha_j + \alpha_j z_s \beta_0 + \varepsilon_{js},$$

where w_{js} is a vector of observed covariates, α_j is teacher j's latent value-added, and z_s denotes student s's observed attributes. In our model (4), δ contains the teacher effects $\{\alpha_j\}$ and γ, x_ℓ contains a teacher dummy and w_{js}, and a_ℓ contains a teacher dummy interacted with the corresponding z_s. In the following, our discussion will be restricted to peer effects models, but one of the advantages of our high-level model (4) is that it can be applied to a wider range of settings, as this example illustrates.

3.1 Inconsistency of least squares

The least squares estimator applied to (4) yields the following estimator of β_0:

$$\hat{\beta}^{\text{NLLS}} = \arg\min_{\beta \in \mathcal{B}} \min_{\delta \in \mathbb{R}^k} \sum_{\ell=1}^{n} \left(y_\ell - x_\ell'\delta - a_\ell'\delta \cdot \beta\right)^2.$$

To give a representation of $\hat{\beta}^{\text{NLLS}}$ that is more amenable to analysis and intuition, we eliminate the nuisance vector δ using the blockwise matrix inversion formula that underpins the Frisch–Waugh–Lovell theorem. To do so, we define the entries of the matrix that residualizes against the regressor $x_\ell + a_\ell\beta$ as $M_{\ell k}(\beta) = \mathbf{1}\{\ell = k\} - (x_\ell + a_\ell\beta)'S(\beta)^{-1}(x_k + a_k\beta)$. We can then represent $\hat{\beta}^{\text{NLLS}}$ as the solution to a minimization problem that does not involve δ:

$$\hat{\beta}^{\text{NLLS}} = \arg\min_{\beta \in \mathcal{B}} \hat{Q}_n(\beta) \qquad \text{where } \hat{Q}_n(\beta) = \sum_{\ell=1}^{n}\sum_{k=1}^{n} M_{\ell k}(\beta) y_\ell y_k.$$

The representation of the least squares estimator as a minimizer of the objective function $\hat{Q}_n$ implies that an almost necessary condition for consistency of $\hat{\beta}^{\text{NLLS}}$ is that the population analog $Q_n(\beta) = \mathbb{E}[\hat{Q}_n(\beta) \mid X, A]$ has a unique minimum at β_0. However, even under the strict exogeneity imposed in Assumption 1, we can write this expectation as the sum of two terms, only the first of which has a unique minimum at β_0. To illustrate this point, let $\sigma_{\ell k} = \mathbb{E}[\varepsilon_\ell \varepsilon_k \mid X, A]$ be the covariance between the ℓ-th

and the k-th error terms and define the part of $a_\ell'\delta$ that provides identifying variation when $\beta_0 = \beta$ as $\tilde{a}_\ell(\beta)'\delta$ where $\tilde{a}_\ell(\beta) = \sum_{k=1}^n M_{\ell k}(\beta)a_k'$. We then have

$$Q_n(\beta) = (\beta - \beta_0)^2 \sum_{\ell=1}^n (\tilde{a}_\ell(\beta)'\delta)^2 + \sum_{\ell=1}^n \sum_{k=1}^n M_{\ell k}(\beta)\sigma_{\ell k}.$$

The full-rank restrictions of Assumption 1 imply that $\sum_{\ell=1}^n (\tilde{a}_\ell(\beta)'\delta)^2 > 0$ for all $\beta \in \mathcal{B}$ so that the first part of Q_n is uniquely minimized at β_0. However, the part of Q_n that involves the error covariances is not, in general, minimized at the truth. The presence of the second part therefore leads to inconsistency of the least squares estimator except in special cases.

Before proceeding to our proposed estimator, it is useful to highlight why the least squares estimator remains consistent with serially uncorrelated and homoscedastic error terms. In this case, we have $\sigma_{\ell k} = \sigma^2 \mathbf{1}\{\ell = k\}$. This property implies that the second part of Q_n simplifies substantially. In fact, this second part becomes independent of β because the matrix function $M = (M_{\ell k})_{\ell,k}$ is a projection onto a linear space of dimension $n - K$, which in turn implies that the sum of the diagonal elements of $M(\beta)$ is $n - K$. We therefore have

$$\sum_{\ell=1}^n \sum_{k=1}^n M_{\ell k}(\beta)\sigma_{\ell k} = \sigma^2 \sum_{\ell=1}^n M_{\ell\ell}(\beta) = \sigma^2(n - K),$$

so that Q_n is uniquely minimized at β_0 in this case. In the special case of the peer effects model (1) without additional control variables w_{it}, this observation was also made by Arcidiacono et al. (2012); Hong and Lattanzio (2025) extends the consistency result under homoskedasticity to the model with controls.

As the regression model in (4) involves an unobserved regressor $a_\ell'\delta$ which is itself estimated from the data, one might be tempted to apply standard measurement-error logic and conjecture that the least squares estimator is attenuated toward zero. As we show in Appendix B.3 for the simple case, the bias in fact has indeterminate sign; least squares estimates available in the literature may differ from the underlying truth in systematic but unknown directions.

3.2 Cross-fit correction to least squares

Our proposed estimator relies on the standard cross-sectional assumption that errors are conditionally independent in model (4). While such an assumption places restrictions on the patterns of dependence that can be allowed for in the data, it does not rule out dependence across observations at a lower level of aggregation, as discussed next.

Assumption 2 (Conditional independence)**.** *Conditional on X and A, $\{\varepsilon_\ell\}_{\ell=1}^n$ are jointly independent.*

Assumption 2 implies that the error variances are of the form $\sigma_{\ell k} = \sigma_\ell^2 \mathbf{1}\{\ell = k\}$. Specifically, we allow for heteroscedasticity in the non-linear regression model in (4).

Remark 2. In the peer effects model in equation (1), there are reasons to be wary of assuming independence among the error terms. For example, it seems reasonable to allow for wage errors to be serially dependent within a particular employment spell. Such dependence can be accommodated by writing down a model as in (1), collapsing the data to the level of employment spells, and then considering the resulting version of model (4) for the collapsed data. Section 5 further illustrates this approach and the biases that can arise from ignoring serial dependence in the microdata. We therefore highlight that, when choosing the particular point estimator used (through the level at which the data is collapsed), the researcher needs to take into account the dependence structure of the error terms. We note, however, that cross-sectional dependence across ε_ℓ when observations share a common peer group or a common employer/classroom is not allowed here, while the inclusion of common shocks as in 6.2.1 potentially relaxes this restriction. Braun and Verdier (2023) allows for such cross-sectional dependence in error terms following a different estimation strategy.

To introduce our proposed estimator, it is useful to describe the inconsistency of least squares in terms of derivatives of the objective functions introduced previously. Viewed through this lens, the least squares estimator is a zero of the sample moment function $\hat{m}_n = \nabla_\beta \hat{Q}_n$, and the source of inconsistency in least squares is that the population analog $m_n = \nabla_\beta Q_n$ is not equal to zero at β_0. Under Assumption 2, the gradient m_n at β_0 deviates from zero by

$$m_n(\beta_0) = \sum_{\ell=1}^{n} \nabla_\beta M_{\ell\ell}(\beta_0)\sigma_\ell^2. \tag{5}$$

Our proposed estimator is a zero of a sample moment function defined as the difference between $\hat{m}_n$ and an estimator of the part that leads to the non-zero expectation in (5). We construct this sample moment using cross-fit, or leave-one-out, estimators of the individual error variances:

$$\hat{\sigma}^2_\ell(\beta) = \frac{y_\ell \hat{\varepsilon}_\ell(\beta)}{M_{\ell\ell}(\beta)} \tag{6}$$

where $\hat{\varepsilon}_\ell(\beta) = y_\ell - (x_\ell + a_\ell\beta)'\hat{\delta}^{\text{LS}}(\beta)$ is the regression residual at β and $\hat{\delta}^{\text{LS}}(\beta) = S(\beta)^{-1}\sum_{\ell=1}^n (x_\ell + a_\ell\beta)y_\ell$ is the corresponding least squares estimator of δ. The fact that $\hat{\sigma}^2_\ell$ is a leave-one-out estimator follows from the equivalent representation

$$\hat{\sigma}^2_\ell(\beta) = y_\ell\left(y_\ell - (x_\ell + a_\ell\beta)'\hat{\delta}^{\text{LS}}_{(\ell)}(\beta)\right),$$

in which $\hat{\delta}^{\text{LS}}_{(\ell)}$ is the least squares estimator of δ applied to the sample that excludes the ℓ-th observation: $\hat{\delta}^{\text{LS}}_{(\ell)}(\beta) = \left(\sum_{k\neq\ell}(x_k + a_k\beta)(x_k + a_k\beta)'\right)^{-1}\sum_{k\neq\ell}(x_k + a_k\beta)y_k$.

We use the leave-one-out individual error variance estimators to recenter the moment function $\hat{m}_n$. This leads to our proposed estimator

$$\hat{\beta}^{\text{CF}} = \arg\operatorname*{zero}_{\beta\in\mathcal{B}} \hat{m}^{\text{CF}}_n(\beta) \qquad \text{where } \hat{m}^{\text{CF}}_n(\beta) = \hat{m}_n(\beta) - \sum_{\ell=1}^n \nabla_\beta M_{\ell\ell}(\beta)\hat{\sigma}^2_\ell(\beta). \tag{7}$$

The recentered moment function is finite-sample unbiased at the truth:

***Proposition* 1** (Finite-sample unbiasedness of the CF moment)**.** *Under Assumptions 1 and 2, if* $\min_{l\leq n} M_{ll}(\beta_0) > 0$*,*

$$\mathbb{E}\Big[\hat{m}^{CF}_n(\beta_0) \mid X, A\Big] = 0.$$

The result follows because the cross-fit estimators $\{\hat{\sigma}^2_\ell(\beta_0)\}_{\ell=1}^n$ are unbiased for their respective error variances, so the recentering exactly removes the bias term in (5). The unbiasedness of $\hat{m}^{\text{CF}}_n$ at β_0 is the cornerstone property that motivates $\hat{\beta}^{\text{CF}}$ as a credible alternative to $\hat{\beta}^{\text{NLLS}}$ without requiring homoskedasticity.

The cross-fit variance estimators $\{\hat{\sigma}^2_\ell\}_{\ell=1}^n$ have previously been used in the linear-regression literature to bias-correct non-linear functions of the least squares estimator and to estimate its variance (Kline et al., 2020a; Anatolyev and Sølvsten, 2020; Matsushita and Otsu, 2019; Mikusheva and Sun, 2020; Jochmans, 2020); the use here

differs in that we consider a non-linear regression and use cross-fitting to bias-correct the least squares estimator itself.

There is a long tradition in econometrics of bias-correcting objective functions, rather than their gradients, in an attempt to ensure that their population counterparts are minimized at the truth (e.g., Han and Phillips, 2006; Hausman et al., 2012). Translating that approach here would suggest a penalized objective function $\hat{Q}_n^{\mathrm{CF}}(\beta) = \hat{Q}_n(\beta) - \sum_{\ell=1}^n M_{\ell\ell}(\beta)\hat{\sigma}_\ell^2(\beta)$; however, this objective is identically zero in β and so cannot yield a consistent estimator of β_0. Recentering the gradient instead is what makes the construction work.

3.3 Consistency

Consistency of $\hat{\beta}^{\mathrm{CF}}$ requires conditions beyond those used for identification and finite-sample unbiasedness. We collect them in the following regularity assumption, which is the non-linear analog of the design conditions imposed in Anatolyev and Sølvsten (2020) for linear regression.

Assumption 3 (Regularity for consistency)**.** *(i)* Bounded leverage:

$$\liminf_{n\to\infty} \inf_{\beta\in\mathcal{B}} \min_{1\le\ell\le n} M_{\ell\ell}(\beta) > 0.$$

(ii) Bounded design: *there exist constants* $C_x, C_a, C_\delta < \infty$ *and* $N < \infty$ *such that, for all* $n \ge N$*,*

$$\max_{\ell\le n} \|x_\ell\|_1 \le C_x, \quad \max_{\ell\le n} \|a_\ell\|_1 \le C_a,$$
$$\|\delta\|_\infty \le C_\delta,$$

and

$$\liminf_{n\to\infty} \inf_{\beta\in\mathcal{B}} \lambda_{\min}\big(n^{-1}S(\beta)\big) > 0.$$

(iii) Restricted dimension: $K/n \to \rho \in [0,1)$ *as* $n \to \infty$*.*

(iv) Strong identification: *for every* $\varepsilon > 0$*,*

$$\liminf_{n\to\infty} \inf_{\beta\in\mathcal{B}:\ |\beta-\beta_0|\ge\varepsilon} n^{-1}|m_n^{\mathrm{CF}}(\beta)| > 0,$$

where $m_n^{\mathrm{CF}}(\beta) = \mathbb{E}[\hat{m}_n^{\mathrm{CF}}(\beta) \mid X, A]$ *is the population counterpart of the cross-fit moment function.*

Part (i) ensures that the leave-one-out variance estimator $\hat{\sigma}_\ell^2 = y_\ell \hat{\varepsilon}_\ell / M_{\ell\ell}$ is well-defined with a bounded denominator; this is the standard limited-mobility-type restriction used in the leave-out literature (Kline et al., 2020a; Anatolyev and Sølvsten, 2020). Part (ii) ensures that the least squares estimator of δ for any $\beta \in \mathcal{B}$ is well-defined and imposes row-sparsity and bounded-coefficient conditions suited to high-dimensional fixed-effect designs. The row-sparsity conditions are satisfied in the peer effects setting when the sizes of peer groups are bounded or grow slowly with n, which is a common feature of sparsely matched teacher-student/worker-firm data. Part (iii) excludes the over-parameterized regime in which K grows as fast as n. Part (iv) is an identification condition that requires $m_n^{\mathrm{CF}}(\beta_0) = 0$ to be a well-separated unique solution to the population moment condition. Roughly speaking, this condition is satisfied when the variation in $\tilde{a}_\ell(\beta)'\delta$ is sufficiently large relative to the terms capturing the sensitivity of $\tilde{a}_\ell(\beta)'\delta$ and $M_{\ell\ell}(\beta)$ to changes in β as β moves away from β_0. Otherwise, there may be another $\tilde{\beta} \neq \beta_0$ where the variation in $\tilde{a}_\ell(\tilde{\beta})'\delta$ is cancelled by the sensitivity terms to the extent that $m_n^{\mathrm{CF}}(\tilde{\beta}) = 0$ as well, so that $\hat{\beta}^{\mathrm{CF}}$ may not be consistent for β_0.

In the peer effects setting, the identifying variation in $\tilde{a}_\ell(\beta)'\delta$ comes from changes in peer group quality that are not fully captured by a linear combination of $x_\ell + a_\ell\beta$. This variation is generated by job mobility and worker turnover, as described in Section 2.2. Assumption 3(iv), in particular, requires such variation to be large enough relative to the sensitivity of the nuisance-parameter estimation and bias correction to the choice of β.

We then have the following consistency result for $\hat{\beta}^{\mathrm{CF}}$:

Theorem 1 (Consistency of $\hat{\beta}^{\mathrm{CF}}$)**.** *Under Assumptions 1, 2, and 3,*

$$\hat{\beta}^{CF} \xrightarrow{p} \beta_0 \quad \text{as } n \to \infty.$$

Our point estimation theory shows that the peer effect β_0 can be consistently estimated using the cross-fit correction approach. One of the key advantages of this approach is that the correction itself does not depend on empirical specifications and thus is practitioner-friendly. This is in contrast to Braun and Verdier (2023)'s

approach for estimating β_0, which requires the availability of instruments for peer-average outcomes $\bar{y}_{(i)t}$ whose availability can be application-dependent.

4 Inference

Hypothesis tests regarding the value of β_0 and confidence intervals for β_0 can be constructed based on a normal approximation to the distribution of $\hat{\beta}^{\mathrm{CF}}$. In this case, inference requires an estimator of the variance $V_n(\beta) = \mathbb{V}\big[\hat{m}_n^{\mathrm{CF}}(\beta) \mid X, A\big]$. This section proposes the variance estimator $\hat{V}_n(\beta)$ and establishes the validity of the resulting inference procedure.

4.1 Variance estimator

To introduce and motivate our proposed variance estimator $\hat{V}_n$, it is useful first to give two separate U-statistic representations of $\hat{m}_n^{\mathrm{CF}}$. The first representation is symmetric in the sense that we write $\hat{m}_n^{\mathrm{CF}} = \sum_{\ell=1}^n \sum_{k\neq\ell} U_{\ell k}^{\mathrm{S}} y_\ell y_k$ and the order of the subscripts on the kernel function $U_{\ell k}^{\mathrm{S}}$ does not matter. This representation immediately follows from the definition of $\hat{Q}_n$ and the formulation of $\{\hat{\sigma}_\ell^2\}_{\ell=1}^n$ given in (6), which yields

$$U_{\ell k}^{\mathrm{S}}(\beta) = \nabla_\beta M_{\ell k}(\beta) - M_{\ell k}(\beta)\big(\nabla_\beta \log M_{\ell\ell}(\beta) + \nabla_\beta \log M_{kk}(\beta)\big)/2.$$

The second representation is asymmetric, i.e., $\hat{m}_n^{\mathrm{CF}} = \sum_{\ell=1}^n \sum_{k\neq\ell} U_{\ell k}^{\mathrm{A}} y_\ell y_k$, where $U_{\ell k}^{\mathrm{A}} \neq U_{k\ell}^{\mathrm{A}}$. To define $U_{\ell k}^{\mathrm{A}}$ and connect the two representations, we rely on a small amount of matrix algebra. The projection M is idempotent, $M = M^2$, and differentiating each entry of this identity yields $\nabla_\beta M = M(\nabla_\beta M) + (\nabla_\beta M)M$. Because the derived matrix identity relates $\nabla_\beta M_{\ell k}$ to the two sums, $\sum_{m=1}^n M_{\ell m}\nabla_\beta M_{mk}$ and $\sum_{m=1}^n M_{km}\nabla_\beta M_{m\ell}$, we can decompose $U_{\ell k}^{\mathrm{S}}$ into $\big(U_{\ell k}^{\mathrm{A}} + U_{k\ell}^{\mathrm{A}}\big)/2$ where

$$U_{\ell k}^{\mathrm{A}}(\beta) = 2\sum_{m=1}^n M_{\ell m}(\beta)\nabla_\beta M_{mk}(\beta) - M_{\ell k}(\beta)\nabla_\beta \log M_{kk}(\beta).$$

The usefulness of providing two U-statistic representations of $\hat{m}_n^{\mathrm{CF}}$ is evident from the following expression, which describes the variance $V_n(\beta_0)$ in terms of both the

symmetric and asymmetric kernel functions:

$$V_n(\beta_0) = 2\mathbb{E}\left[\sum_{\ell=1}^{n}\left(\sum_{k\neq\ell} U^{\mathrm{S}}_{\ell k}(\beta_0)y_k\right)\left(\sum_{m\neq\ell} U^{\mathrm{A}}_{\ell m}(\beta_0)y_m\right)\sigma^2_\ell \,|\, X, A\right] - \mathbb{E}\Big[\hat{m}^{\mathrm{CF}}_n(\beta_0)\Big]^2. \quad (8)$$

The squared expectation of the sample moment function, which is included at the end of this equation, is zero. It is nevertheless included here to motivate why our variance estimator subtracts $\big(\hat{m}^{\mathrm{CF}}_n(\beta)\big)^2$ whenever it is evaluated at a β where the sample moment is non-zero.

Our proposed variance estimator drops the expectations present in (8) and replaces the unknown individual error variances $\{\sigma^2_\ell\}_{\ell=1}^n$ with cross-fit analogs. However, because outcome variables already enter the expression in (8), the use of leave-one-out cross-fit estimators may not suffice for consistent variance estimation. We therefore rely on leave-three-out estimators in the construction of $\hat{V}_n$. Specifically, we use

$$\hat{\sigma}^2_{\ell,-km}(\beta) = y_\ell\big(y_\ell - (x_\ell + a_\ell\beta)'\hat{\delta}_{(\ell km)}(\beta)\big) \quad (9)$$

where the leave-three-out estimator of δ is $\hat{\delta}_{(\ell km)}(\beta) = (\sum_{s\neq\ell,k,m}(x_s + a_s\beta)(x_s + a_s\beta)')^{-1}\sum_{s\neq\ell,k,m}(x_s + a_s\beta)y_s$.[4] Our proposed variance estimator is then

$$\hat{V}_n(\beta) = 2\sum_{\ell=1}^{n}\sum_{k\neq\ell}\sum_{m\neq\ell} U^{\mathrm{S}}_{\ell k}(\beta)U^{\mathrm{A}}_{\ell m}(\beta)y_k y_m\hat{\sigma}^2_{\ell,-km}(\beta) - \Big(\hat{m}^{\mathrm{CF}}_n(\beta)\Big)^2. \quad (10)$$

This estimator has the following key property in our setting because the leave-three-out variance estimator $\hat{\sigma}^2_{\ell,-km}(\beta_0)$ is unbiased for σ^2_ℓ and independent of the outcome variables y_k and y_m:

***Proposition* 2** (Unbiasedness of leave-three-out cross-fit estimators)**.** *Under Assumptions 1 and 2, if $\hat{\sigma}^2_{l,-km}(\beta_0)$ is well-defined for all l, $k \neq l$, and $m \neq l$,*

$$\mathbb{E}\Big[\hat{V}_n(\beta_0) \mid X, A\Big] = V_n(\beta_0).$$

[4] When k is equal to m, $\hat{\delta}_{(\ell kk)}(\beta)$ is a leave-two-out estimator since it only drops observations ℓ and k. For this reason, we also use $\hat{\sigma}^2_{\ell,-k}$ when describing $\hat{\sigma}^2_{\ell,-kk}$.

4.2 Consistency and Asymptotic Normality

The consistency of $\hat{V}_n$ and the asymptotic normality of $\hat{\beta}^{\mathrm{CF}}$ require additional moment and variance-estimator regularity conditions. Let $\mathcal{N}_0$ be a fixed neighborhood of β_0 contained in $\mathcal{B}$. Define the leave-two-out and leave-three-out determinants for distinct observations:

$$D_{\ell k}(\beta) = \begin{vmatrix} M_{\ell\ell}(\beta) & M_{\ell k}(\beta) \\ M_{\ell k}(\beta) & M_{kk}(\beta) \end{vmatrix},$$

$$D_{\ell k m}(\beta) = \begin{vmatrix} M_{\ell\ell}(\beta) & M_{\ell k}(\beta) & M_{\ell m}(\beta) \\ M_{\ell k}(\beta) & M_{kk}(\beta) & M_{km}(\beta) \\ M_{\ell m}(\beta) & M_{km}(\beta) & M_{mm}(\beta) \end{vmatrix},$$

and $D_{kk}(\beta) = 0$ and $D_{\ell kk}(\beta) = D_{\ell k}(\beta)$ for $\ell \neq k$ by convention, where $|\cdot|$ denotes the determinant of a matrix.

Assumption 4 (Regularity for inference)**.** *(i)* Bounded eighth moments: *there exist constants $C_8 < \infty$ and $N < \infty$ such that, for all $n \geq N$,*

$$\max_{\ell \leq n} \mathbb{E}[|\varepsilon_\ell|^8 \mid X, A] \leq C_8.$$

(ii) Leave-two-out and leave-three-out conditioning:

$$\liminf_{n\to\infty} \inf_{\beta \in \mathcal{N}_0} \min_{\ell,k,m:\, k,m \neq \ell} D_{\ell k m}(\beta) > 0.$$

Part (i) strengthens the corresponding condition in Assumption 1 to account for higher-order interactions of the error terms reflected in the formula for $\hat{V}_n(\beta_0)$. Part (ii) requires that the leave-two-out estimators $\sigma^2_{(lk)}$ and the leave-three-out estimators $\sigma^2_{(lkm)}$ are well-defined uniformly over n and locally around β_0. See Section 4.3 for a discussion of when this condition does not hold.

The variance estimator $\hat{V}_n$ is consistent, and when combined with the consistency of $\hat{\beta}^{\mathrm{CF}}$ (Theorem 1), we obtain asymptotic normality for the studentized estimator. The following theorem summarizes this result, justifying a standard inference procedure for β_0:

Theorem 2 (Asymptotic inference)**.** *Suppose Assumptions 1, 2, 3, and 4 hold, and*

that

$$\liminf_{n\to\infty} n^{-1}V_n(\beta_0) > 0 \qquad \textit{and} \qquad \liminf_{n\to\infty} \inf_{\beta\in\mathcal{N}_0} n^{-1}|\nabla_\beta m_n^{\mathrm{CF}}(\beta)| > 0.$$

Then:

(i) [Variance estimator consistency] $\hat{V}_n(\beta_0)/V_n(\beta_0) \xrightarrow{p} 1$ *and* $\hat{V}_n(\hat{\beta}^{\mathrm{CF}})/V_n(\beta_0) \xrightarrow{p} 1$ *as* $n \to \infty$.

(ii) [Studentized asymptotic normality]

$$\frac{\nabla_\beta \hat{m}_n^{\mathrm{CF}}(\hat{\beta}^{\mathrm{CF}})}{\sqrt{\hat{V}_n(\hat{\beta}^{\mathrm{CF}})}}\big(\hat{\beta}^{\mathrm{CF}} - \beta_0\big) \xrightarrow{d} N(0,1)$$

as $n \to \infty$.

Remark 3. In the applied literature that relies on the non-linear least squares estimator $\hat{\beta}^{\mathrm{NLLS}}$, the standard practice for inference is to rely on bootstrapping (e.g., Arcidiacono et al., 2012; Cornelissen et al., 2017). Currently, there is no theoretical justification for the use of the bootstrap in settings like ours, and our simulations reported in Section 5 suggest that the wild bootstrap may fail to yield valid inference. Our proposed variance estimator $\hat{V}_n$, on the other hand, provides a viable alternative to the bootstrap, and Theorem 2 provides the theoretical justification for valid inference when $\sqrt{\hat{V}_n(\hat{\beta}^{\mathrm{CF}})}/\nabla_\beta \hat{m}_n^{\mathrm{CF}}(\hat{\beta}^{\mathrm{CF}})$ is used as the standard error for $\hat{\beta}^{\mathrm{CF}}$.

4.3 Practical considerations

As with the leave-one-out estimator for σ_l^2 (6), the leave-three-out estimator (9) can be computed without explicitly constructing leave-three-out samples; the following representations can be leveraged for computation. As shown in Anatolyev and Sølvsten (2020), we have

$$\begin{aligned} &y_l - (x_l + a_l\beta)'\hat{\delta}_{(lk)}(\beta) \\ &= \frac{M_{kk}(\beta)(y_l - (x_l + a_l\beta)'\hat{\delta}^{\mathrm{LS}}(\beta)) - M_{lk}(\beta)(y_k - (x_k + a_k\beta)'\hat{\delta}^{\mathrm{LS}}(\beta))}{D_{lk}(\beta)}, \end{aligned}$$

and

$$
\begin{aligned}
&y_l - (x_l + a_l\beta)'\hat{\delta}_{(lkm)}(\beta) \\
&= \frac{(y_l - (x_l + a_l\beta)'\hat{\delta}^{\mathrm{LS}}(\beta))}{D_{lkm}(\beta)/D_{km}(\beta)} \\
&- \frac{M_{lk}(\beta)(y_k - (x_k + a_k\beta)'\hat{\delta}_{(lk)}(\beta)) + M_{lm}(\beta)(y_m - (x_m + a_m\beta)'\hat{\delta}_{(lm)}(\beta))}{D_{lkm}(\beta)/D_{km}(\beta)}.
\end{aligned}
$$

We can use these representations to compute $\hat{\sigma}_{\ell,-km}(\beta)$ and therefore $\hat{V}_n(\beta)$ more efficiently than by directly constructing leave-three-out samples when the number of observations is moderate, with n on the order of a few tens of thousands.

However, if the number of observations is considerably larger, exact computation of $\hat{V}_n$ appears to be infeasible. For this reason, we introduce a recursive representation of the product $y_k y_m \hat{\sigma}^2_{\ell,-km}$, which we truncate to approximate $\hat{V}_n$. Defining $r_{\ell k} = M_{\ell k}/\sqrt{M_{\ell\ell}M_{kk}}$, which is bounded by one in absolute value, we can write

$$
\begin{aligned}
y_k y_m \hat{\sigma}^2_{\ell,-km} = &\underbrace{y_k y_m \hat{\sigma}^2_{\ell} - M_{\ell k}\frac{y_\ell y_m \hat{\sigma}^2_k}{M_{\ell\ell}} - \left(M_{\ell m} - \frac{M_{\ell k}M_{km}}{M_{kk}}\right)\frac{y_\ell y_k \hat{\sigma}^2_m}{M_{\ell\ell}}}_{=\Upsilon_{\ell km}} \\
&+ (r^2_{\ell k} + r^2_{\ell m} - r_{\ell k}r_{\ell m}r_{km}) y_k y_m \hat{\sigma}^2_{\ell,-km} + \left(M_{\ell m} - \frac{M_{\ell k}M_{km}}{M_{kk}}\right) M_{km}\frac{y_\ell y_m \hat{\sigma}^2_{k,-\ell m}}{M_{\ell\ell}M_{mm}}.
\end{aligned}
$$

The truncated approximation $\hat{V}_n^{(tr)}$ is then given by replacing $y_k y_m \hat{\sigma}^2_{\ell,-km}$ with $\Upsilon_{\ell km}$ in the expression for $\hat{V}_n$:

$$
\hat{V}_n^{(tr)}(\beta) = 2\sum_{\ell=1}^{n}\sum_{k\neq\ell}\sum_{m\neq\ell} U^{\mathrm{S}}_{\ell k}(\beta)U^{\mathrm{A}}_{\ell m}(\beta)\Upsilon_{\ell km}(\beta) - \left(\hat{m}_n^{\mathrm{CF}}(\beta)\right)^2.
$$

As shown in Section A, $\hat{V}_n^{(tr)}$ can be expressed in matrix form and computed more efficiently than the exact $\hat{V}_n$. We note, however, that the consistency of $\hat{V}_n^{(tr)}$ is not guaranteed in general, and we therefore recommend using $\hat{V}_n$ whenever computationally feasible. We leave the theoretical analysis of the consistency of $\hat{V}_n^{(tr)}$ to future work.

Finally, we note that some of the leave-three-out estimators $\hat{\sigma}^2_{\ell,-km}$ may be ill-defined or unstable when $D_{lkm}(\beta)$ is close to zero, i.e., when Assumption 4 (ii) is violated. It is still possible to conduct valid inference when such (l,k,m) combinations

are not too common, either by (i) dropping them from the sample or (ii) replacing $\hat{\sigma}^2_{\ell,-km}$ with a conservative estimate. See Section D for a formal treatment of the second approach.

5 Benchmark application: revisiting Arcidiacono et al. (2012) classroom peer effects

We illustrate our method by replicating and extending the seminal application of Arcidiacono et al. (2012), who estimated classroom peer effects using student transcript data. We use the universal transcript data from the University of Wisconsin–Madison, with a sharp natural experiment: in Spring 2020, the COVID-19 pandemic forced all undergraduate teaching online, plausibly eliminating the in-classroom peer interactions that the canonical model captures. This section contains the application; Section 6 reports a second, complementary application to workplace peer effects using the Veneto matched employer–employee panel. We also provide Monte Carlo simulations calibrated to the UW Spring 2019 design that compare the non-linear least squares estimator with wild-bootstrap standard errors against our cross-fit estimator with the proposed standard errors.

5.1 Sample selection and econometric specification

We use the administrative student-level records from the Registrar's Office at the University of Wisconsin-Madison, which has a universal coverage of all the students. The database contains multiple records, including demographics, high school test scores, and transcript information.

Sample selection We focus on all the courses that are where the students are all undergraduate students. Under the UW system, it means all the courses with code under 300, e.g., Econ 101, because the university allows graduate students to take courses with code above 300. Given the university has a large body of graduate students, we exclude such courses to avoid noisy interactions coming from graduate students. We keep students who have valid A–F grade information for a given course. We assign numeric grade equivalents to the letter grades following the university GPA

system: A = 4, AB = 3.5, B = 3, BC = 2.5, C = 2, D = 1, and F = 0.[5]

In particular, we define the *peer group* as all the students in the same discussion section in the same course. Given that undergraduate courses are typically large in class sizes, the university typically assigns multiple teaching assistants to hold discussion sections for each course. The discussion section typically has fewer than 20 students and allows students to interact and discuss problems with the guidance of the teaching assistant.

We are particularly interested in the semester when the Covid-19 pandemic hit the university during the Spring semester of the academic year 2019/2020. Because the university decided to shift all the undergraduate courses entirely online. Given this situation is unprecedented and online learning tools are new to every student, we expect that the peer effect from student interactions may be largely reduced, if not disappeared. In fact, this is widely observed and discussed among teaching assistants and professors during the semester that nearly all students have turned off their cameras during the discussion section, and classroom interaction almost vanished. We also explore the Spring semester of the academic year 2018/2019 as our placebo semester, as the courses offered for these two semesters are almost identical.

Econometric specification We use the following regression specification following equation (1), which is similar to the specification in Arcidiacono et al. (2012).

$$y_{ij} = \alpha_i + \bar{\alpha}_{(i)j} \cdot \beta + \psi_j + \varepsilon_{ij}, \tag{11}$$

where α_i is the student fixed effect, which measures the ability of a student. We define ψ_j as a course-professor pair fixed effect. For example, if Econ 101 is taught by three professors, Alan, Bob, and Cathy, we define them as three different courses. The reason is straightforward: each professor typically makes their own syllabi and exams. The peer quality $\bar{\alpha}_{(i)j}$ is defined as the average students' ability within the same peer group, excluding the student i.

Note that we do not include the time dimension in this specification because we

[5] Arcidiacono et al. (2012) uses a similar database from the University of Maryland from 1999 to 2001 – a much older period than what we focus on. They also divide their main sample into three categories: humanities, social science, and math and science, according to the official course types. The UW system does not have a similar corresponding classification system. As our main purpose is to demonstrate the ideas of our proposed estimator and inference, we keep the sample selection simple by pooling all the students together.

estimate each semester separately. One can still identify the student fixed effects using data from one semester because students must take at least one course to maintain their full-time student status. Also, each student chooses different lists of courses every semester, and the mobility across different courses is massive, which is important as it serves as the key identifying variation for the peer effect β_0.

An extension that adds an endogenous peer effect through peer outcomes (in the spirit of Bramoullé et al., 2009; Bramoullé, Djebbari and Fortin, 2020) is provided in Appendix E as an empirical robustness exercise. Because the formal asymptotic theory for the joint estimator extends beyond the scope of this paper, we keep the main text focused on (11).

5.2 Peer effects estimates

Table 1 reports estimates from equation (11) in the Spring semesters of 2019 and 2020, comparing the non-linear least squares (NLLS) estimator of Arcidiacono et al. (2012) with our cross-fit (CF) estimator.

Let us first compare the results in the Spring semester of 2019 when the Covid-19 pandemic has not taken place. We find the non-linear least squared (NLLS) estimate for β is around 0.25, and the wild bootstrapping standard error is around 0.03. These figures are in a similar range to what was found in Arcidiacono et al. (2012) although they use a sample from a different university during a much older period. Our proposed cross-fit (CF) estimate is 0.17, which is around 30 percent smaller than the NLLS estimate. Our proposed standard error is 0.033, which is 10 percent larger than the wild bootstrap standard error. The results suggest there is a large bias correction using our method, which is both economically and statistically meaningful. As discussed above, the variance of the plug-in fixed effects can be biased, $\hat{\sigma}_{\bar{\alpha}_{(i)t}}$ is estimated to be much smaller using the technique adopted from Kline, Saggio and Sølvsten (2020b). As a result, the one-standard-deviation effect is 0.052 under NLLS and plug-in estimator of $\sigma_{\bar{\alpha}_{(i)t}}$ while the counterpart effect is 0.032 under our method, which is about 39 percent smaller.

We now turn to the Spring 2020 semester, when the COVID-19 pandemic shifted all undergraduate teaching online. As we conjectured above, social interactions during discussion sections were sharply reduced, so the classroom peer effect should fall substantially, if not vanish. The NLLS estimate suggests that a small but statistically

Table 1: UW-Madison register data for spring semesters in 2019 and 2020

	Spring 2019		Spring 2020	
	NLLS	CF	NLLS	CF
$\hat{\beta}$	0.249	0.169	0.049	−0.002
	(0.030)	(0.033)	(0.026)	(0.033)
$\hat{\sigma}_{\bar{\alpha}_{(i)t}}$ plug-in	0.210		0.157	
$\hat{\sigma}_{\bar{\alpha}_{(i)t}}$ KSS		0.188		0.143
1-sd effect	0.052	0.032	0.008	−0.000

Notes: Wild bootstrap standard errors for NLLS. The proposed standard errors based on $\hat{V}_n$ (leave-three-out approximation) for cross-fit. The full table including the endogenous-peer-effect extension appears in Appendix E.

significant positive effect remains: a one-standard-deviation increase in peer ability is associated with a 0.8% rise in the course grade. Using our method, the CF estimate is close to zero with a point estimate of −0.002 and a standard error of 0.033, statistically insignificant. The implied one-standard-deviation effect is essentially zero.

5.3 Simulation exercises

The application demonstrates the empirical bite of the bias correction in a single sample. The next step is to verify that the gap between $\hat{\beta}^{\text{NLLS}}$ and $\hat{\beta}^{\text{CF}}$ is a systematic consequence of heteroskedasticity. In this subsection, we conduct a Monte Carlo exercise calibrated to the Spring 2019 specification (11) to compare the two estimators across 1,000 simulated draws.

The calibration takes the CF point estimate $\beta_0 = 0.169$ as the data-generating parameter, with the fitted values $(\hat{\alpha}, \hat{\psi}_j)$ from the CF estimation supplying the linear component of the regression. Errors are drawn from heteroskedastic normal distributions whose variances are calibrated to the squared residuals from the CF fit. For each draw, we re-estimate both $\hat{\beta}^{\text{NLLS}}$ (with wild-bootstrap standard errors, following Arcidiacono et al., 2012 and Cornelissen et al., 2017) and $\hat{\beta}^{\text{CF}}$ (with the proposed standard errors based on $\hat{V}_n$), and record three metrics in Table 2: (i) bias of the point estimator relative to β_0, (ii) ratio of the average standard error to the realized standard deviation of estimates across draws, and (iii) coverage of the 95% nominal confidence interval.

Table 2: Simulations using UW-Madison 2019 spring semester

	NLLS	CF
Point estimator:		
Bias	0.041	−0.005
Standard deviation	0.028	0.026
Bias/SD	1.440	−0.183
Standard error:		
Standard error/SD - 1	−19.3%	0.3%
Coverage:		
Nominal 95% CI	56.5%	92.9%

Notes: Wild bootstrap standard errors for NLLS. Approximate leave-three-out standard errors for cross-fit.

Point estimator. The NLLS bias of 0.041 corresponds to a bias-to-SD ratio of 1.44: under heteroskedasticity, the NLLS estimator has an order-one bias relative to its sampling variability. The CF estimator delivers a bias of −0.005 (bias-to-SD ratio −0.18), consistent with the finite-sample-unbiasedness of $\hat{m}_n^{\mathrm{CF}}(\beta_0)$ in Proposition 1 and the consistency in Theorem 1. The two point estimators have similar realized standard deviations (0.028 vs. 0.026), so the bias-to-SD difference is driven by bias rather than by efficiency.

Standard error. The average wild-bootstrap standard error for NLLS is 19.3% smaller than the realized standard deviation of $\hat{\beta}^{\mathrm{NLLS}}$ across draws. The wild bootstrap is standard practice in the applied literature but lacks theoretical justification in non-linear models like (4) (see also Bickel and Freedman, 1983), and it systematically under-states sampling variability in the heteroskedastic design simulated here. The proposed standard error based on $\hat{V}_n$ tracks the realized standard deviation of $\hat{\beta}^{\mathrm{CF}}$ almost exactly (0.3% deviation), consistent with the unbiasedness of $\hat{V}_n$ established in Proposition 2.

Coverage. The 95% confidence interval based on NLLS + wild bootstrap covers β_0 in only 56.5% of draws — a substantial under-coverage combining the NLLS bias with the under-stated standard error. The confidence interval based on $\hat{\beta}^{\mathrm{CF}}$ and $\hat{V}_n$ covers in 92.9% of draws, close to the 95% nominal target and supporting the use of our procedure for inference in heteroskedastic peer-effects designs.

6 Workplace peer effects

This section applies the cross-fit estimator to the Veneto matched employer–employee panel, a leading data setting in the AKM literature (Abowd et al., 1999b; Card et al., 2018). The application has two purposes. The first is to document the empirical bite of the bias correction at scale on the data setting that the workplace peer-effects literature has used for two decades (e.g., Mas and Moretti, 2009; Cornelissen et al., 2017; Hong and Lattanzio, 2025). The second is to extend the canonical two-way worker–firm wage decomposition to admit average coworker quality as a third component. Consistent estimation of the peer coefficient, delivered by the cross-fit correction, is what makes this extended decomposition operational. Section 6.1 describes the data and sample; Section 6.2 reports specification and peer-effect estimates; Section 6.3 reports the variance decomposition.

6.1 Data and sample

We use the Veneto Worker History (VWH) database, an administrative matched employer–employee panel covering the entire private-sector workforce in the Veneto region of northern Italy. The database is constructed from social-security records reported to the Italian National Institute of Social Security (INPS) and contains accurate annual earnings without top-coding, weeks worked, occupation (manager, white-collar, blue-collar, apprentice), and contract type. We restrict attention to the years 1995–2001, a window over which the Veneto labor market was in a steady environment with nearly full employment (Tattara and Valentini, 2010; Serafinelli, 2019).[6]

We apply the standard sample restrictions in the AKM literature: keep each worker's primary job in each year, restrict to workers aged 16–65, drop part-time and apprentice contracts, drop firms with more than 5,000 employees, and require peer groups with at least two workers. To support both the cross-fit estimator and the bias-corrected variance estimator of Kline et al. (2020b) used in Section 6.3, we further restrict to the leave-one-out connected set of firms.[7] The resulting sample

[6] We follow the standard practice in the AKM literature of restricting to a moderately long but not exceedingly long panel; this lets us observe enough worker mobility for identification while limiting the time over which worker fixed effects must be treated as fixed.

[7] The largest connected set is the largest set of firms linked by worker mobility, which is required to identify worker and firm fixed effects (Abowd et al., 1999b). The leave-one-out connected set

contains approximately five million person-year observations covering more than one million workers and around seventy thousand firms.

Peer groups are defined at the firm × broad occupation × year level, where broad occupation refers to one of three professional categories (blue-collar, white-collar, manager).[8] Additional sample-selection details, descriptive statistics, mobility rates by firm size, and the institutional setting of the Italian labor market over the sample window are deferred to Appendix F.

6.2 Peer-effect estimates

We estimate the wage equation

$$y_{it} = \alpha_i + \beta \cdot \bar{\alpha}_{(i)t} + \psi_{j(i,t)} + w_{it}'\gamma + \varepsilon_{it}, \tag{12}$$

where y_{it} is the log weekly wage, α_i is the worker fixed effect, $\psi_{j(i,t)}$ is the firm fixed effect, $\bar{\alpha}_{(i)t}$ is the average peer quality in worker i's peer group, and w_{it} contains age-squared (normalized to age minus 40), tenure, tenure squared, log firm size, occupation indicators, and year indicators. The specification is exactly (3) in Section 2, applied to the VWH sample.

Identification of β in (12) draws on two complementary sources of mobility-induced variation in $\bar{\alpha}_{(i)t}$. Job switchers contribute variation through changes in their peer group; job stayers contribute variation through the entry and exit of coworkers around them. Both are quantitatively important in our sample: approximately 8% of workers change firms in a given year, and the average firm replaces around 20% of its workforce annually, with this turnover rate remaining above 15% even among firms with more than 1,000 employees (see Appendix F for the firm-size profile). These mobility statistics support the strong-identification condition of Assumption 3(iv),

strengthens this condition by requiring connectedness to survive the removal of any single mover, and is what is needed to apply the bias correction of Kline et al. (2020b). The leave-one-out sample is similar to the full sample in observable characteristics; see Appendix F for a comparison.

[8] Defining peer groups requires balancing breadth (capturing potential coworker interactions) against narrowness (ensuring within-group interaction is plausible). Portugal et al. (2024) document that workplace peer-effect estimates are quantitatively similar across alternative occupation definitions using comparable Portuguese matched data; in Appendix F we report that reassigning managers to their pre-promotion occupation (blue- or white-collar) leaves our estimates essentially unchanged. To the extent that the peer-group definition imperfectly captures the true pattern of interaction in the firm, the resulting measurement error attenuates our estimates (Cornelissen et al., 2017; Nix, 2016), so our reported coefficients can be viewed as a lower bound on the true peer effect.

which requires that mobility-induced variation in peer-group composition translates into non-trivial variation in the unobserved $\bar{\alpha}_{(i)t}$.

We estimate (12) by the cross-fit estimator $\hat{\beta}^{\text{CF}}$ developed in Section 3. For comparison, we also report the iterated NLLS estimator $\hat{\beta}^{\text{NLLS}}$ of Arcidiacono et al. (2012). Following Kline et al. (2020b), we bias-correct the variance of the peer-quality regressor $\hat{\sigma}_{\bar{\alpha}_{(i)t}}$ so that the implied one-standard-deviation effect is comparable across specifications. Table 3 reports the baseline estimates in Column (1).

The cross-fit estimator returns $\hat{\beta}^{\text{CF}} = 0.413$ with the proposed standard error 0.035, statistically significant at the 1% level. The bias-corrected standard deviation of average peer quality is $\hat{\sigma}_{\bar{\alpha}_{(i)t}} = 0.191$, implying that a one-standard-deviation increase in coworker quality raises a worker's wage by 7.8%, an effect of magnitude comparable to the return to one year of schooling in Italy over the same period (Lucifora, Comi and Brunello, 2000). The iterated NLLS estimator returns $\hat{\beta}^{\text{NLLS}} = 0.351$, about 15% smaller than the cross-fit estimate. The classroom application in Section 5 gives the opposite case — in Spring 2019 the NLLS estimate is 0.249 against a cross-fit estimate of 0.169, so NLLS overstates the peer effect by about 50% — while here NLLS understates it by about 15%. Our workplace point estimate lies in the upper part of the range reported in comparable matched employer–employee studies (Cornelissen et al., 2017; Hong and Lattanzio, 2025), with quantitative differences across settings plausibly reflecting genuine variation in workplace interaction across labor markets.

6.2.1 Common shocks

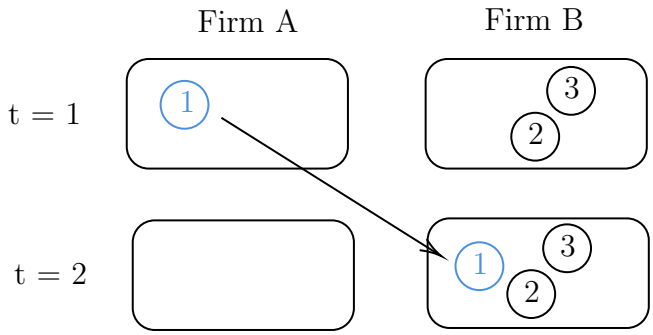
A potential concern with (12) is that peer-group-specific time-varying shocks may correlate with shocks to peer quality. For example, a firm may adopt new technology specific to one occupation, simultaneously raising wages and (estimated) worker quality within that occupation. To address this, we re-estimate (12) replacing $\psi_{j(i,t)}$ together with the occupation and year fixed effects by a saturated firm $\times$ occupation $\times$ year fixed effect $\xi_{j(i,t),o(i),t}$. The peer effect remains identified in this specification: $\bar{\alpha}_{(i)t}$ excludes worker i and varies within a firm-occupation-year cell as the composition and size of the peer group change. Figure 1 illustrates the source of identifying variation that survives in this specification: when worker 1 moves into firm B, the stayer worker 2 experiences a change in average peer quality from α_3 to $\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_3$ even though the firm-occupation-year fixed effect is unchanged.

Table 3: Workplace peer effects in Veneto

	Baseline (1)	Common shocks (2)	Placebo peers (3)
$\hat{\beta}^{\text{CF}}$	0.413	0.367	0.016
	(0.035)	(0.028)	(0.005)
$\hat{\sigma}_{\bar{\alpha}_{(i)t}}$ (KSS)	0.191	0.198	0.210
1-SD effect	7.8%	7.3%	0.3%
Worker FE	x	x	x
Firm FE	x		x
Occupation FE, year FE, controls	x		x
Firm × Occupation × Year FE		x	

Notes: Column (1) is the baseline specification. Column (2) replaces firm, occupation, and year fixed effects with a saturated firm × occupation × year fixed effect (peer-group fixed effect) to absorb peer-group-specific common shocks. Column (3) defines the placebo peer group as workers in *other* occupations within the same firm-year.

Figure 1: Identifying variation under firm × occupation × year fixed effects



Notes: Depiction of three workers (denoted 1, 2, and 3) and worker 1's mobility between two firms (denoted A and B). In the first period, workers 2 and 3 are peers in firm B; in the second period, workers 1, 2, and 3 are all in firm B. Worker compositions in firm A other than worker 1 are omitted. For the stayer (worker 2), peer composition changes from $\{3\}$ to $\{1, 3\}$, generating identifying variation for β that is not absorbed by the firm-occupation-year fixed effect.

Column (2) of Table 3 reports $\hat{\beta}^{\text{CF}} = 0.367$, broadly similar to the baseline. We retain (12) as our preferred specification because it uses both job-mover and job-stayer variation and preserves the link to the canonical AKM decomposition that we will exploit in Section 6.3.

6.2.2 Placebo peers

If the estimated peer effect reflects coworker influence rather than firm- or occupation-level common shocks not absorbed by the controls, then re-defining the peer group as workers in *other* occupations within the same firm-year should return an effect close to zero. Column (3) of Table 3 reports $\hat{\beta}^{\text{CF}} = 0.016$ under this placebo definition, implying a one-standard-deviation effect of 0.3%. The placebo coefficient is two orders of magnitude smaller than the baseline, supporting the interpretation that the baseline estimate captures within-occupation peer influence rather than common variation across occupations within the firm.

6.3 Variance decomposition

The consistent peer-effect estimate from (12) allows us to extend the canonical AKM wage-variance decomposition to admit average coworker quality as a third component. Following Card et al. (2018) and Sorkin (2018), we use the "ensemble" decomposition,

$$\text{Var}(y_{it}) = \text{Cov}(y_{it}, \alpha_i) + \text{Cov}(y_{it}, \psi_{j(i,t)}) + \text{Cov}(y_{it}, \beta \cdot \bar{\alpha}_{(i)t}) + \text{Cov}(y_{it}, w_{it}'\gamma) + \text{Cov}(y_{it}, \varepsilon_{it}), \tag{13}$$

which assigns to each component the share of wage variance it explains in the regression. The covariance terms between α_i and $\psi_{j(i,t)}$ and between α_i and $\bar{\alpha}_{(i)t}$, after appropriate expansion of (13), identify worker–firm sorting and worker–coworker sorting respectively. As is well known, plugging fixed-effect estimates directly into a variance decomposition produces sizable bias under limited mobility (Andrews et al., 2008; Kline et al., 2020b). We apply the bias-correction technique of Kline et al. (2020b) to all variance and covariance components in (13); in our setting this is the same correction that delivers the bias-corrected $\hat{\sigma}_{\bar{\alpha}_{(i)t}}$ used in Section 6.2.

Table 4 reports the decomposition. Column (1) shows the canonical AKM decomposition (omitting average coworker quality) as a benchmark; Column (2) shows the decomposition from the peer model in (12). In the AKM model, worker fixed effects explain 53.3% of wage variance and firm fixed effects explain 18%, with worker–firm sorting accounting for an additional 10.8% (correlation 0.21). These magnitudes are in line with what the AKM literature has documented across European and U.S. labor markets (e.g., Card et al., 2018; Sorkin, 2018).

Two findings emerge from the peer-model column. First, average coworker quality

Table 4: Wage variance decomposition: AKM versus peer model

Share of $\mathrm{Var}(y_{it})$ *explained by*	AKM model (1)	Peer model (2)
Worker effects, α_i	53.3%	47.1%
Firm effects, $\psi_{j(i,t)}$	18.0%	13.0%
Coworker effects, $\beta \cdot \bar{\alpha}_{(i)t}$		11.0%
Sorting: $2\,\mathrm{Cov}(\alpha_i, \psi_{j(i,t)})$	10.8%	3.4%
Sorting: $2\,\mathrm{Cov}(\alpha_i, \beta \cdot \bar{\alpha}_{(i)t})$		27.6%
Correlation: (worker, firm)	0.211	0.080
Correlation: (worker, coworker)		0.596

Notes: Ensemble decomposition of $\mathrm{Var}(y_{it})$ following Card et al. (2018). Variance and covariance components are bias-corrected following Kline et al. (2020b). Column (1) is the canonical AKM specification without $\bar{\alpha}_{(i)t}$; Column (2) is the peer model in (12) with $\hat{\beta}^{\mathrm{CF}}$ as the peer coefficient. The control-variate and residual rows are omitted.

explains 11% of wage variance — a share comparable to the 13% explained by firm effects in the same specification. The coworker channel is therefore as economically large as the firm channel, but it is invisible to the two-way model.[9] Portugal et al. (2024) report a smaller coworker share in Portuguese matched data; the difference is consistent with the bias-corrected estimator used here yielding a larger consistent estimate of β than the homoskedastic NLLS estimator used in their work.

Second, the structure of labor-market sorting shifts substantially once coworkers are admitted. The worker–firm sorting correlation that has anchored the AKM literature falls from 0.21 to 0.08, while a worker–coworker sorting correlation of 0.60 emerges as the dominant sorting margin in the data. The worker–coworker correlation is consistent in magnitude with what Lopes de Melo (2018) reports in Brazilian data using an indirect plug-in approach. The reading we favor is not that the AKM literature has been measuring sorting incorrectly, but that what the two-way model attributes to worker–firm sorting is, in substantial part, worker–coworker sorting that the model has no language for. With a consistent peer-effect estimate in hand, the decomposition can separate the two for the first time.

[9]The interpretation that the firm channel "shrinks" from 18% in the AKM model to 13% in the peer model is suggestive but not literal: the AKM and peer models are different statistical models, and the comparison reflects what each can attribute to firms when coworker quality is or is not separately identified. The clean reading is that the peer model identifies an 11% coworker channel of comparable economic size to firms, which the AKM specification cannot separate.

7 Conclusion

This paper develops estimation and inference for panel-data peer-effects models in which social influence operates through an unobserved individual characteristic and the errors are heteroskedastic. The non-linear least squares estimator that anchors this literature is inconsistent under heteroskedasticity, and the sign of its bias is governed by the empirical design rather than by any general principle. Our cross-fit correction removes this bias: it yields a moment that is unbiased in finite samples at the truth, an estimator that is consistent under standard regularity conditions, and an analytic standard error built from a leave-three-out variance estimator. The stochastic approximations in Appendix A make the procedure feasible on administrative-scale data.

The two applications show that the correction is more than a theoretical refinement and that the direction of the bias genuinely cannot be signed in advance. In the University of Wisconsin transcript data, the cross-fit estimate is about 30% below NLLS in a pre-pandemic benchmark, and during the Spring 2020 online semester — when in-person classroom interaction was largely absent — NLLS still reports a positive and significant peer effect while our estimate is close to zero and statistically insignificant. In the Italian matched employer–employee panel, by contrast, the cross-fit estimate is about 15% *larger* than the iterated NLLS benchmark. The same correction thus moves the estimate down in one setting and up in the other, exactly as the indeterminate-sign result predicts; Monte Carlo experiments calibrated to the classroom design confirm that the gap is a systematic consequence of heteroskedasticity rather than a coincidence.

A consistent peer coefficient also makes possible a wage-variance decomposition that the canonical two-way model cannot deliver. Extending the AKM decomposition to incorporate average coworker quality, we find that coworkers account for a share of wage variance comparable to that of firms, and that the dominant axis of sorting in the labor market runs between workers and their coworkers rather than between workers and firms. To our knowledge this is the first decomposition to separate these two sorting channels using a consistent estimate of the peer coefficient, and it suggests that a meaningful part of what two-way models attribute to firms in fact reflects who a worker's colleagues are.

Finally, the method is not specific to peer averaging: it applies to non-linear

least squares problems in which a low-dimensional parameter multiplies a latent individual effect, a structure shared by models of heterogeneous teacher value-added. Appendix E sketches a descriptive extension that allows peer outcomes to enter directly; a formal asymptotic treatment of the joint (β, λ) estimator under Bramoullé et al. (2009)-type network identification remains an open problem that we leave to future work.

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Appendix A Computation on large datasets

This appendix spells out the algebraical details needed for implementation of the proposed point estimator and confidence set. We first provide matrix representations of the moment function $\hat{m}_n^{\mathrm{CF}}$, its derivative $\nabla_\beta \hat{m}_n^{\mathrm{CF}}$, and the truncated variance estimator $\hat{V}_n^{(tr)}$. We then introduce stochastic approximations to those quantities that make computation feasible at scale.

To facilitate a description of $\hat{m}_n^{\mathrm{CF}}$, $\nabla_\beta \hat{m}_n^{\mathrm{CF}}$, and $\hat{V}_n^{(tr)}$ that uses matrix algebra, define $y = (y_1, \ldots, y_n)'$, $X = (x_1, \ldots, x_n)'$, $A = (a_1, \ldots, a_n)'$, $\sigma^2 = (\sigma_1^2, \ldots, \sigma_n^2)'$, $\hat{\sigma}^2 = (\hat{\sigma}_1^2, \ldots, \hat{\sigma}_n^2)'$ and $M^{(d)} = (M_{11}, \ldots, M_{nn})'$. Note that $\hat{\sigma}^2$ and $M^{(d)}$ are functions of β and recall that $S(\beta) = (X + A\beta)'(X + A\beta)$ while $\hat{\varepsilon}(\beta) = M(\beta)y$ and $\hat{\delta}^{\mathrm{LS}}(\beta) = S(\beta)^{-1}(X + A\beta)y$ are the residuals and estimated coefficients from a linear regression of y on $X + A\beta$. For any vector v, $\mathrm{diag}[v]$ is the diagonal matrix with v along its main diagonal. Elementwise products and ratios are denoted by $\odot$ and $\oslash$, respectively.

A.1 Matrix representations

Upon utilizing the matrix derivative relationship $\nabla(F^{-1}) = -F^{-1}(\nabla F)F^{-1}$, we find that $\nabla_\beta M = -(D + D')$ for the nilpotent matrix $D(\beta) = M(\beta)AS(\beta)^{-1}(X + A\beta)'$. Letting $\Lambda(\beta) = \mathrm{diag}\big[\nabla_\beta \log\big(M^{(d)}(\beta)\big)\big]$, we can then write $\hat{m}_n^{\mathrm{CF}} = -2\hat{\varepsilon}' A\hat{\delta}^{\mathrm{LS}} - \hat{\varepsilon}'\Lambda y$. Similarly, we have the representation

$$\begin{aligned}\nabla_\beta \hat{m}_n^{\mathrm{CF}} = 2\Big(\|MA\hat{\delta}^{\mathrm{LS}}\|^2 + \|A\hat{\delta}^{\mathrm{LS}}\|^2 - \|A\hat{\delta}^{\mathrm{LS}} - D'y\|^2\Big) \\ + (MA\hat{\delta}^{\mathrm{LS}} + D'y)'\Lambda y - \hat{\varepsilon}'(\nabla_\beta \Lambda)y.\end{aligned}$$

After collection of the asymmetric kernel weights $U_{\ell k}^{\mathrm{A}}$ in the matrix $U^{\mathrm{A}} = \{U_{\ell k}^{\mathrm{A}}\}_{\ell,k}$, we have the representation $U^{\mathrm{A}} = -(2D + M\Lambda)$ where $\hat{m}_n^{\mathrm{CF}} = y'U^{\mathrm{A}}y$. Similarly, the matrix of symmetric kernel weights is $U^{\mathrm{S}} = (U^{\mathrm{A}} + U^{\mathrm{A}\prime})/2$ where again $\hat{m}_n^{\mathrm{CF}} = y'U^{\mathrm{S}}y$.

We then have

$$\begin{aligned}\hat{V}_n^{(tr)}/2 = {} & y'U^{\mathrm{S}}\mathrm{diag}\big[\hat{\sigma}^2\big]U^{\mathrm{A}}y - \Big(\hat{m}_n^{\mathrm{CF}}\Big)^2/2 \\ & - \mathrm{trace}\Big(\mathrm{diag}\big[\hat{\sigma}^2\big]M\mathrm{diag}\big[U^{\mathrm{A}}y \odot y \oslash M^{(d)}\big]U^{\mathrm{S}}\Big) \\ & - \mathrm{trace}\Big(\mathrm{diag}\big[\hat{\sigma}^2\big]M\mathrm{diag}\big[U^{\mathrm{S}}y \odot y \oslash M^{(d)}\big]U^{\mathrm{A}}\Big) \\ & + \mathrm{trace}\Big(\mathrm{diag}\big[\hat{\sigma}^2\big]M\mathrm{diag}\big[y \oslash M^{(d)}\big]\big(U^{\mathrm{S}} \odot M\big)\mathrm{diag}\big[y \oslash M^{(d)}\big]U^{\mathrm{A}}\Big).\end{aligned}$$

A.2 Stochastic approximations

Exact computation of the log-derivative matrix functions Λ and $\nabla_\beta\Lambda$ appearing in $\hat{m}_n^{\mathrm{CF}}$, U^{A}, U^{S}, and $\nabla_\beta\hat{m}_n^{\mathrm{CF}}$ is challenging in large samples as it typically requires evaluation and storage of the two $n \times n$ matrix functions D and M. We therefore rely on a stochastic approximation to the numerical derivative. Exact evaluation of the three matrix traces appearing in $\hat{V}_n^{(tr)}$ is similarly challenging, and we therefore rely on a related stochastic approximation to those traces. For these approximations, we let p be a large even integer and ϵ a small positive real. Our implementation of the approximation uses $p = 200$ and $\epsilon = 0.005$. Furthermore, we let $(r_1, \ldots, r_p) \in \mathbb{R}^{n\times p}$ be a random matrix with *i.i.d.* Rademacher entries (discrete uniform random variables with support $\{-1, 1\}$).

Our stochastic approximation to the vector $M^{(d)}$ is[10]

$$\check{M}^{(d)} = \Big\{\sum\nolimits_{s=1}^{p}(Mr_s \odot Mr_s)\Big\} \oslash \Big\{\sum\nolimits_{s=1}^{p}(Mr_s \odot Mr_s) + (Pr_s \odot Pr_s)\Big\}$$

where $P = I - M$. Each entry in the sums that enter $\check{M}^{(d)}(\beta)$ are squares of the residuals and fitted values in a regression of r_s on $X + A\beta$.

Remark 4. The approximation to $M^{(d)}$ is motivated by the following mean relationships for the numerator and denominator in $\check{M}^{(d)}$: $\mathbb{E}[\sum_{s=1}^{p}(Mr_s \odot Mr_s)] = pM^{(d)}$ and $\mathbb{E}[\sum_{s=1}^{p}(Mr_s \odot Mr_s) + (Pr_s \odot Pr_s)] = p\mathbf{1}_n$ where $\mathbf{1}_n = (1, \ldots, 1)' \in \mathbb{R}^n$. A related version of $\check{M}^{(d)}$ that instead uses the non-random denominator p was suggested in Achlioptas (2003) in the spirit of Johnson and Lindenstrauss (1984); see also Kline et al. (2020a). $\check{M}^{(d)}$ improves on that version by enforcing the shape constraints that the entries of $M^{(d)}$ has support on $[0, 1]$. See also Kline, Saggio and Sølvsten (2021)

[10] For any observations where the entries of $\check{M}^{(d)}$ are below 0.01, we replace them by 0.01.

for a derivation of $\check{M}^{(d)}$ as a feasible version of a minimum variance combination of two separate stochastic approximations to $\check{M}^{(d)}$.

The approximation to Λ is based on a finite difference and $\check{M}^{(d)}$:

$$\check{\Lambda}(\beta) = \epsilon^{-1}\text{diag}\Big[\log\Big(\check{M}^{(d)}(\beta+\epsilon) \oslash \check{M}^{(d)}(\beta)\Big)\Big].$$

The approximation to $\nabla_\beta \Lambda$ is similarly

$$\check{\nabla}_\beta \check{\Lambda} = \epsilon^{-1}(\check{\Lambda}(\beta+\epsilon) - \check{\Lambda}(\beta))$$

All simulations and empirical results use the stochastic approximations $\check{m}_n^{\text{CF}} = -2\hat{\varepsilon}' A\hat{\delta} - \hat{\varepsilon}'\check{\Lambda} y$, $\check{U}^{\text{A}} = -(2D + M\check{\Lambda})$, $\check{U}^{\text{S}} = (\check{U}^{\text{A}} + \check{U}^{\text{A}\prime})/2$, $\check{\sigma}^2 = y \odot \hat{\varepsilon} \oslash \check{M}^{(d)}$,

$$\begin{aligned}\check{\nabla}_\beta \check{m}_n^{\text{CF}} = 2\Big(\|MA\hat{\delta}^{\text{LS}}\|^2 + \|A\hat{\delta}^{\text{LS}}\|^2 - \|A\hat{\delta}^{\text{LS}} - D'y\|^2\Big) \\ + (MA\hat{\delta}^{\text{LS}} + D'y)'\check{\Lambda} y - \hat{\varepsilon}'(\check{\nabla}_\beta \check{\Lambda}) y,\end{aligned}$$

and the stochastic approximation to $\hat{V}_n^{(tr)}$:

$$\begin{aligned}\check{V}_n^{(tr)}/2 = {} & y'\check{U}^{\text{S}}\text{diag}[\check{\sigma}^2]\check{U}^{\text{A}} y - \Big(\check{m}_n^{\text{CF}}\Big)^2 \\ & - \tfrac{1}{p}\sum\nolimits_{s=1}^{p}\Big(M\big(\check{\sigma}^2 \odot r_s\big)\odot \check{U}^{\text{A}} y \odot y \oslash \check{M}^{(d)}\Big)'\check{U}^{\text{S}} r_s \\ & - \tfrac{1}{p}\sum\nolimits_{s=1}^{p}\Big(M\big(\check{\sigma}^2 \odot r_s\big)\odot \check{U}^{\text{S}} y \odot y \oslash \check{M}^{(d)}\Big)'\check{U}^{\text{A}} r_s \\ & + \tfrac{1}{p}\sum\nolimits_{s=1}^{p/2}\Big(M\big(\check{\sigma}^2 \odot r_s\big)\odot r_{p/2+s} \odot y \oslash \check{M}^{(d)}\Big)' M\Big(\check{U}^{\text{A}} r_s \odot \check{U}^{\text{S}} r_{p/2+s} \odot y \oslash \check{M}^{(d)}\Big) \\ & + \tfrac{1}{p}\sum\nolimits_{s=1}^{p/2}\Big(M\big(\check{\sigma}^2 \odot r_{p/2+s}\big)\odot r_s \odot y \oslash \check{M}^{(d)}\Big)' M\Big(\check{U}^{\text{A}} r_{p/2+s} \odot \check{U}^{\text{S}} r_s \odot y \oslash \check{M}^{(d)}\Big).\end{aligned}$$

Remark 5. Our approximations to the first two traces in $\hat{V}_n^{(tr)}$ are called Hutchinson approximations (Hutchinson, 1989). They utilize that an unbiased estimator for the trace of a matrix F is the quadratic form $r_1' F r_1$ and that this quadratic form is easy to evaluate numerically. For the third trace entering $\hat{V}_n^{(tr)}$ the relevant quadratic form for use with the Hutchinson approximation is numerically challenging to evaluate due to the matrix Hadamard product $U^{\text{S}} \odot M$. For this trace, we combine the Hutchinson approximation with "sample splitting" in the sense that we utilize $(r_1 \odot r_2)'(F_1 r_1 \odot F_2 r_2)$ as an unbiased estimator for the trace of $F_1 \odot F_2$.

Remark 6. The computationally most demanding part of evaluating $\check{m}_n^{\text{CF}}(\beta)$ and

$\check{V}_n^{(tr)}(\beta)$ are to find the solutions to linear systems of equations of the kind $S(\beta)x = b$ for various values of b. To construct $\check{m}_n^{\mathrm{CF}}(\beta)$ there are $2p+1$ such systems to solve while there is an additional $2 \cdot (2p+1)$ systems involved in computing $\check{V}_n^{(tr)}(\beta)$. The time it takes to solve those systems of equations depends in large part on the number of regressors present in the model.[11]

Appendix B Derivations from the simple example

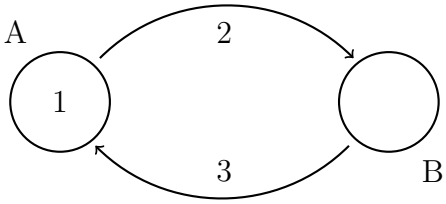
As a service to the reader we quickly state the main definitions used throughout the paper and proofs. Here y, X, and A stacks the observations for y_ℓ, x_ℓ', and a_ℓ' and σ^2 stacks the individual error variances σ_ℓ^2. Furthermore, $R(\beta) = X + A\beta$, $S = R'R$, $P = RS^{-1}R'$, $M = I - P$, $M^{(d)}$ is the diagonal of M, $\Lambda = \operatorname{diag}\left[\nabla_\beta \log\left(M^{(d)}\right)\right]$, $D = MAS^{-1}R'$, $U^{\mathrm{A}} = -(2D + M\Lambda)$, and $U^{\mathrm{S}} = (U^{\mathrm{A}} + U^{\mathrm{A}\prime})/2$. The non-linear least squares estimator is $(\hat{\beta}^{\mathrm{NLLS}}, \hat{\delta}^{\mathrm{LS}}) = \arg\min_{\beta \in \mathcal{B}, \delta \in \mathbb{R}^k} \sum_{\ell=1}^n \left(y_\ell - x_\ell'\delta - a_\ell'\delta \cdot \beta\right)^2$ while the key objective and moments functions are $\hat{Q}_n = y'My$, $\hat{m}_n = \nabla_\beta Q_n$, $\hat{m}_n^{\mathrm{CF}} = \hat{m}_n - y'M\Lambda y$, $Q_n = \mathbb{E}[\hat{Q}_n \mid X, A]$, and $m_n^{\mathrm{CF}} = \mathbb{E}[\hat{m}_n^{\mathrm{CF}} \mid A, X]$. Finally, $\mathbf{1} = (1, \ldots, 1)' \in \mathbb{R}^n$.

B.1 Motivating example: three individuals, two firms

To illustrate that the mobility-induced variation in average peer quality can lead to point identification, we consider a special case of (1). This special case also serves to highlight how the least squares estimator relies on homoskedasticity while our proposed cross-fit estimator does not. Suppose now that $w_{it}'\gamma$ is equal to $\psi_{j(i,t)}$, which was introduced in (3), that the time horizons T_i are all equal to two, that the error terms are independent across time and individuals, and that the data is generated in triplets of individuals where the first individual *stays* with the same employer for both periods, while the other two individuals *move* between two triplet-specific firms as depicted in Figure 1. The data for a generic triplet is then governed by the following six equations

$$
\begin{aligned}
y_{11} &= \alpha_1 + \alpha_2\beta_0 + \psi_A + \varepsilon_{11}, & y_{12} &= \alpha_1 + \alpha_3\beta_0 + \psi_A + \varepsilon_{12},\\
y_{21} &= \alpha_2 + \alpha_1\beta_0 + \psi_A + \varepsilon_{21}, & y_{22} &= \alpha_2 + \psi_B + \varepsilon_{22},\\
y_{31} &= \alpha_3 + \psi_B + \varepsilon_{31}, & y_{32} &= \alpha_3 + \alpha_1\beta_0 + \psi_A + \varepsilon_{32}.
\end{aligned}
$$

[11] It also depends on the structure of the model through the degree of sparsity in S.



Appendix Figure 1: Depiction of three individuals (denoted 1, 2, and 3) and their mobility among two firms (denoted A and B). In the first period, individuals 1 and 2 are peers, while individuals 1 and 3 are peers in the second period. In both periods, firm A has two employees and firm B has one employee.

The only part of this data that contains information about β_0 is a first difference for the stayer, $\mathcal{Y} = y_{12} - y_{11}$, and two differences involving both the movers, $\mathcal{X} = y_{32} - y_{21}$ and $\mathcal{Z} = y_{31} - y_{22}$. We can view $\mathcal{Y}$ as an outcome in a regression model that includes the unobserved regressor $\alpha_3 - \alpha_2$,

$$\mathcal{Y} = (\alpha_3 - \alpha_2)\beta_0 + (\varepsilon_{12} - \varepsilon_{11}),$$

while $\mathcal{X}$ and $\mathcal{Z}$ are independent noisy measurements of the unobserved regressor,

$$\mathcal{X} = \alpha_3 - \alpha_2 + (\varepsilon_{32} - \varepsilon_{21}) \quad \text{and} \quad \mathcal{Z} = \alpha_3 - \alpha_2 + (\varepsilon_{31} - \varepsilon_{22}).$$

It is therefore immediate that a necessary and sufficient condition for point identification of β_0 is that $\alpha_3 - \alpha_2$ is not zero across all the triplets in the data. Unless the data is segregated into homogeneous groups, such identifying variation will be present.

A particularly simple estimator of β_0 can be constructed from $(\mathcal{Y}, \mathcal{X}, \mathcal{Z})$ by averaging the two instrumental variables estimators that let $\mathcal{X}$ and $\mathcal{Z}$ take turns as noisy regressor and instrument. Equivalently, this estimator is the sample analog of the moment condition

$$\beta_0 = \frac{1}{2}\frac{\mathbb{E}[\mathcal{Z}\mathcal{Y}]}{\mathbb{E}[\mathcal{Z}\mathcal{X}]} + \frac{1}{2}\frac{\mathbb{E}[\mathcal{X}\mathcal{Y}]}{\mathbb{E}[\mathcal{X}\mathcal{Z}]}. \tag{14}$$

This simple estimator is a special case of the cross-fit estimator introduced in Section 3.

To highlight the differences and similarities between the cross-fit and least squares

estimators, we first re-express the denominator of (14) so that

$$\beta_0 = \frac{\mathbb{E}[(\mathcal{Z}+\mathcal{X})\mathcal{Y}]}{\mathbb{E}[\mathcal{Z}^2+\mathcal{X}^2]-\mathbb{E}[(\mathcal{Z}-\mathcal{X})^2]}. \tag{15}$$

If the error terms are homoskedastic so that the unexplained variance in $\mathcal{Y}$ is the same as in $\mathcal{X}$ and $\mathcal{Z}$, we can alternatively express the variance $\mathbb{E}[(\mathcal{Z}-\mathcal{X})^2]$ appearing in the denominator of (15) as $4\sigma^2(\beta_0)$ where $\sigma^2(\beta)$ is a variance function that draws on both the movers and the stayer:

$$\sigma^2(\beta) = \frac{\mathbb{E}[(\mathcal{Z}-\mathcal{X})^2+(\mathcal{Y}-\mathcal{X}\beta)^2+(\mathcal{Y}-\mathcal{Z}\beta)^2]}{4(2+\beta^2)}. \tag{16}$$

The non-linear least squares estimator minimizes the sample analog of (16). Additionally, we can describe the least squares estimator as a solution to the sample analog of a first order condition for minimization of (16), which looks exactly like (15), except that it uses $4\sigma^2(\beta)$ instead of $\mathbb{E}[(\mathcal{Z}-\mathcal{X})^2]$:

$$\beta = \frac{\mathbb{E}[(\mathcal{Z}+\mathcal{X})\mathcal{Y}]}{\mathbb{E}[\mathcal{Z}^2+\mathcal{X}^2]-4\sigma^2(\beta)}. \tag{17}$$

In the presence of heteroskedasticity, it is problematic to rely on an estimator that solves the sample analog of (17) as $4\sigma^2(\beta_0)$ will then be different from $\mathbb{E}[(\mathcal{Z}-\mathcal{X})^2]$ and this will in turn lead to an inconsistent estimator. In that case, the least squares estimator will be amplified (attenuated) relative to the truth if $\mathcal{Y}$ has higher (lower) unexplained variance than $\mathcal{X}$ and $\mathcal{Z}$. In this example, a natural assertion would be that the log-wage difference for the single stayer, $\mathcal{Y}$, has lower unexplained variance than the two log-wage differences involving the two movers, $\mathcal{X}$ and $\mathcal{Z}$. If that assertion holds true, the least squares estimator will understate the magnitude of the peer effects.

The sample analogs of (14)–(17) are special cases of the general formulas in Section 3; the explicit connection is developed in Section B.4 below.

The structure exposed by (14) is the textbook measurement-error correction: $\mathcal{X}$ and $\mathcal{Z}$ are two independent noisy measurements of the same latent regressor $\alpha_3-\alpha_2$, and each serves as instrument for the other. Section 3 explains how the general cross-fit moment $\hat{m}_n^{\mathrm{CF}}$ in (7) extends this IV construction to a high-dimensional non-linear regression: leave-one-out projection plays the role of the "second noisy measurement,"

and the recentering algebraically combines the two views as the IV moment in (14) does.

B.2 Information content of (Y, X, Z) in the simple example

Suppose that data is generated according to the setup of Section 2.2. We use the sufficiency principle together with an added assumption that the error terms are homoskedastic normal to argue that $(\mathcal{Y}, \mathcal{X}, \mathcal{Z})$ contains all the information about β_0. Removing this assumption of homoskedastic normality (without adding other assumptions) can never lead the discarded data to become informative. The original data can be recovered from $(\mathcal{Y}, \mathcal{X}, \mathcal{Z})$ and $(\tilde{\mathcal{Y}}, \tilde{\mathcal{X}}, \tilde{\mathcal{Z}})$ where $\tilde{\mathcal{Y}} = y_{12}+y_{11}$, $\tilde{\mathcal{X}} = y_{32}+y_{21}$, and $\tilde{\mathcal{Z}} = y_{31} + y_{22}$. Furthermore, it follows from standard variance calculations that $(\mathcal{Y}, \mathcal{X}, \mathcal{Z})$ and $(\tilde{\mathcal{Y}}, \tilde{\mathcal{X}}, \tilde{\mathcal{Z}})$ are independent (conditionally on explanatory variables). Finally, the mean of $(\mathcal{Y}, \mathcal{X}, \mathcal{Z})$ depends only on β_0 and $\alpha_3 - \alpha_2$, and even when those two parameters are known, the mean of $(\tilde{\mathcal{Y}}, \tilde{\mathcal{X}}, \tilde{\mathcal{Z}})$ is unrestricted in $\mathbb{R}^3$. It therefore follows that $(\tilde{\mathcal{Y}}, \tilde{\mathcal{X}}, \tilde{\mathcal{Z}})$ only contains information about its mean, or in other words, that $(\mathcal{Y}, \mathcal{X}, \mathcal{Z})$ is sufficient for β_0, $\alpha_3 - \alpha_2$, and the unknown error variance.

B.3 Bias direction of the least squares estimator in the simple example

We now show that the least squares estimator is amplified (attenuated) relative to the truth if $\mathcal{Y}$ has higher (lower) unexplained variance than $\mathcal{X}$ and $\mathcal{Z}$. Towards this end, define $(\sigma^2_{\mathcal{Y}}, \sigma^2_{\mathcal{X}}, \sigma^2_{\mathcal{Z}})$ as the unexplained variance of $(\tilde{\mathcal{Y}}, \tilde{\mathcal{X}}, \tilde{\mathcal{Z}})$. Furthermore, suppose that β^* is the unique global minimizer of $\sigma^2(\beta)$ and that the least squares estimator converge in probability to β^*. We can write

$$4\sigma^2(\beta) = \frac{2(\beta - \beta_0)^2\mathbb{E}[(\alpha_3 - \alpha_2)^2] + 2\sigma^2_{\mathcal{Y}} + \mathbb{E}[(\mathcal{Z} - \mathcal{X})^2](1 + \beta^2)}{2 + \beta^2}.$$

If $\sigma^2_{\mathcal{Y}} \leq \min\{\sigma^2_{\mathcal{X}}, \sigma^2_{\mathcal{Z}}\}$, then we have the ordering $4\sigma^2(\beta^*) \leq 4\sigma^2(\beta_0) \leq \mathbb{E}[(\mathcal{Z} -$

$\mathcal{X})^2] < \mathbb{E}[\mathcal{Z}^2 + \mathcal{X}^2]$ which together with the first order condition in (17) yields

$$|\beta^*| = \frac{|\mathbb{E}[(\mathcal{Z}+\mathcal{X})\mathcal{Y}]|}{\mathbb{E}[\mathcal{Z}^2+\mathcal{X}^2] - 4\sigma^2(\beta^*)} \leq \frac{|\mathbb{E}[(\mathcal{Z}+\mathcal{X})\mathcal{Y}]|}{\mathbb{E}[\mathcal{Z}^2+\mathcal{X}^2] - 4\sigma^2(\beta_0)}$$
$$\leq \frac{|\mathbb{E}[(\mathcal{Z}+\mathcal{X})\mathcal{Y}]|}{\mathbb{E}[\mathcal{Z}^2+\mathcal{X}^2] - \mathbb{E}[(\mathcal{Z}-\mathcal{X})^2]} = |\beta_0|.$$

If instead we have $\sigma_{\mathcal{Y}}^2 \geq \max\{\sigma_1^2, \sigma_2^2\}$, then we have $\mathbb{E}[\mathcal{Z}^2+\mathcal{X}^2] = \lim_{|\beta|\to\infty} 4\sigma^2(\beta) > 4\sigma^2(\beta^*) \geq \mathbb{E}[(\mathcal{Z}-\mathcal{X})^2]$ and in turn

$$|\beta^*| = \frac{|\mathbb{E}[(\mathcal{Z}+\mathcal{X})\mathcal{Y}]|}{\mathbb{E}[\mathcal{Z}^2+\mathcal{X}^2] - 4\sigma^2(\beta^*)} \geq \frac{|\mathbb{E}[(\mathcal{Z}+\mathcal{X})\mathcal{Y}]|}{\mathbb{E}[\mathcal{Z}^2+\mathcal{X}^2] - \mathbb{E}[(\mathcal{Z}-\mathcal{X})^2]} = |\beta_0|.$$

B.4 Connecting the simple example to the general formulas

Next, we connect the expressions in (14)–(17) with the general formulas introduced in Section 3. As the model is over parameterized, we drop ψ_B from the model so that the design has full rank. For the six observations in a given triplet, the corresponding part of the matrix $M(\beta)$ is

$$I - \frac{1}{2(2+\beta^2)} \begin{bmatrix} 2(1+\beta^2) & 2 & \beta & \beta & -\beta & -\beta \\ 2 & 2(1+\beta^2) & -\beta & -\beta & \beta & \beta \\ \beta & -\beta & 3+\beta^2 & 1 & -1 & 1+\beta^2 \\ \beta & -\beta & 1 & 3+\beta^2 & 1+\beta^2 & -1 \\ -\beta & \beta & -1 & 1+\beta^2 & 3+\beta^2 & 1 \\ -\beta & \beta & 1+\beta^2 & -1 & 1 & 3+\beta^2 \end{bmatrix}$$

and in analogy with the sufficiency argument above, the formulation in terms of the full data leads to one half times the objective defined for $(\mathcal{Y}, \mathcal{X}, \mathcal{Z})$ where the corresponding matrix $M(\beta)$ is

$$I - \frac{1}{2+\beta^2} \begin{bmatrix} \beta^2 & \beta & \beta \\ \beta & 1 & 1 \\ \beta & 1 & 1 \end{bmatrix} = \frac{1}{2+\beta^2} \begin{bmatrix} 2 & -\beta & -\beta \\ -\beta & 1+\beta^2 & -1 \\ -\beta & -1 & 1+\beta^2 \end{bmatrix}.$$

Thus the contribution of any particular triplet to the least squares objective function is four times the sample analog of (16) since

$$\mathcal{Y}^2+\mathcal{X}^2+\mathcal{Z}^2-\frac{(\mathcal{Y}\beta+\mathcal{X}+\mathcal{Z})^2}{2+\beta^2}=\frac{(\mathcal{Z}-\mathcal{X})^2+(\mathcal{Y}-\mathcal{X}\beta)^2+(\mathcal{Y}-\mathcal{Z}\beta)^2}{2+\beta^2}.$$

The cross-fit objective function contribution defined for a single triplet $(\mathcal{Y},\mathcal{X},\mathcal{Z})$ at (β_1,β) is similarly found as

$$\frac{-2\mathcal{Y}(\mathcal{Z}+\mathcal{X})\beta_1-2\mathcal{Z}\mathcal{X}}{2+\beta_1^2}+\frac{(1+\frac{1+\beta_1^2}{1+\beta^2})\mathcal{Y}(\mathcal{Z}+\mathcal{X})\beta+2\frac{1+\beta_1^2}{1+\beta^2}\mathcal{Z}\mathcal{X}}{2+\beta_1^2}.$$

Therefore, each triplet contributes $(-\mathcal{Y}(\mathcal{Z}+\mathcal{X})+2\mathcal{Z}\mathcal{X}\beta)\frac{2}{(2+\beta^2)(1+\beta^2)}$ to the first order condition for a minimum of the cross-fit objective in β_1 at $\beta_1=\beta$. Thus, we obtain the sample analog of the first order condition in (14).

Appendix C Proofs

This appendix collects formal proofs of the theorems and propositions stated in the main text.

C.1 Proof of Theorem 1

Throughout this proof all expectations and stochastic orders are conditional on (X,A). Let

$$m_n^{\mathrm{CF}}(\beta)=\mathbb{E}[\hat{m}_n^{\mathrm{CF}}(\beta)\mid X,A].$$

By Proposition 1, $m_n^{\mathrm{CF}}(\beta_0)=0$. The argument is a Z-estimator consistency argument for the triangular array of moments.

Lemma 1 (Zero of a uniformly convergent moment)**.** *Suppose that, for every $\varepsilon>0$,*

$$\liminf_n \inf_{\beta\in\mathcal{B}:|\beta-\beta_0|\geq\varepsilon} n^{-1}|m_n^{\mathrm{CF}}(\beta)|>0,$$

and

$$\sup_{\beta\in\mathcal{B}} n^{-1}|\hat{m}_n^{\mathrm{CF}}(\beta)-m_n^{\mathrm{CF}}(\beta)|\xrightarrow{p}0.$$

Then any exact zero $\hat{\beta}^{\mathrm{CF}}$ *of* $\hat{m}_n^{\mathrm{CF}}$ *in* $\mathcal{B}$ *satisfies* $\hat{\beta}^{\mathrm{CF}} \xrightarrow{p} \beta_0$.

Proof. Fix $\varepsilon > 0$. By the separation condition, there are $\eta_\varepsilon > 0$ and N_ε such that, for $n \geq N_\varepsilon$,

$$\inf_{\beta \in \mathcal{B}: |\beta - \beta_0| \geq \varepsilon} n^{-1} |m_n^{\mathrm{CF}}(\beta)| \geq \eta_\varepsilon.$$

On the event $\{|\hat{\beta}^{\mathrm{CF}} - \beta_0| \geq \varepsilon\}$,

$$\begin{aligned} \eta_\varepsilon &\leq n^{-1} |m_n^{\mathrm{CF}}(\hat{\beta}^{\mathrm{CF}})| \\ &\leq \sup_{\beta \in \mathcal{B}} n^{-1} |\hat{m}_n^{\mathrm{CF}}(\beta) - m_n^{\mathrm{CF}}(\beta)| + n^{-1} |\hat{m}_n^{\mathrm{CF}}(\hat{\beta}^{\mathrm{CF}})|. \end{aligned}$$

The last term is zero by definition of $\hat{\beta}^{\mathrm{CF}}$, and the first term is $o_p(1)$. Hence the probability of $\{|\hat{\beta}^{\mathrm{CF}} - \beta_0| \geq \varepsilon\}$ converges to zero. □

Assumption 3(iv) is the separation condition in Lemma 1. It remains to prove the uniform convergence condition.

For interpretation of this separation condition, define

$$q_\ell(\beta) = \tilde{a}_\ell(\beta)' \delta, \qquad R_n(\beta) = 2 \sum_{\ell=1}^{n} q_\ell(\beta) \nabla_\beta q_\ell(\beta),$$

and

$$B_n(\beta) = \sum_{\ell=1}^{n} \frac{\nabla_\beta M_{\ell\ell}(\beta)}{M_{\ell\ell}(\beta)} \mu_\ell q_\ell(\beta), \qquad \mu_\ell = \mathbb{E}[y_\ell \mid X, A].$$

Using

$$\mathbb{E}[\hat{\sigma}_\ell^2(\beta) \mid X, A] = \sigma_\ell^2 + (\beta_0 - \beta) \frac{\mu_\ell q_\ell(\beta)}{M_{\ell\ell}(\beta)},$$

the population CF moment can be written as

$$m_n^{\mathrm{CF}}(\beta) = (\beta - \beta_0) \left[2 \sum_{\ell=1}^{n} q_\ell(\beta)^2 + (\beta - \beta_0) R_n(\beta) + B_n(\beta) \right].$$

Thus Assumption 3(iv) requires the leading identifying variation in $q_\ell(\beta)$ not to be cancelled by the sensitivity terms $R_n(\beta)$ and $B_n(\beta)$ outside any neighborhood of β_0.

Lemma 2 (Uniform norm bounds)**.** *Under Assumption 3(i)–(iii),*

$$\|A\|_2 = O(\sqrt{n}), \qquad \sup_{\beta\in\mathcal{B}} \|R(\beta)\|_2 = O(\sqrt{n}), \qquad \sup_{\beta\in\mathcal{B}} \|S(\beta)^{-1}\|_2 = O(n^{-1}),$$

$$\sup_{\beta\in\mathcal{B}} \|D(\beta)\|_2 = O(1), \qquad \sup_{\beta\in\mathcal{B}} \|D(\beta)\|_F = O(\sqrt{n}),$$

$$\sup_{\beta\in\mathcal{B}} \|\nabla_\beta M(\beta)\|_2 = O(1), \qquad \sup_{\beta\in\mathcal{B}} \|\nabla_\beta D(\beta)\|_2 = O(1),$$

$$\sup_{\beta\in\mathcal{B}} \|\Lambda(\beta)\|_\infty = O(1), \qquad \sup_{\beta\in\mathcal{B}} \|\nabla_\beta \Lambda(\beta)\|_\infty = O(1),$$

$$\sup_{\beta\in\mathcal{B}} \|U^{\mathrm{A}}(\beta)\|_2 = O(1), \qquad \sup_{\beta\in\mathcal{B}} \|\nabla_\beta U^{\mathrm{A}}(\beta)\|_2 = O(1), \qquad \sup_{\beta\in\mathcal{B}} \|U^{\mathrm{A}}(\beta)\|_F = O(\sqrt{n}),$$

$$\sup_{\beta\in\mathcal{B}} \|\nabla_\beta U^{\mathrm{A}}(\beta)\|_F = O(\sqrt{n}), \qquad \sup_{\beta\in\mathcal{B}} \|\nabla_{\beta\beta} U^{\mathrm{A}}(\beta)\|_2 = O(1).$$

Proof. Bounded row ℓ_1 norms imply bounded row ℓ_2 norms. Hence

$$\|A\|_2 \le \|A\|_F = \left(\sum_{\ell=1}^{n} \|a_\ell\|_2^2\right)^{1/2} \le C_a\sqrt{n} = O(\sqrt{n}).$$

The same argument gives $\|X\|_2 = O(\sqrt{n})$, and compactness of $\mathcal{B}$ gives

$$\sup_{\beta\in\mathcal{B}} \|R(\beta)\|_2 \le \|X\|_2 + \sup_{\beta\in\mathcal{B}} |\beta| \|A\|_2 = O(\sqrt{n}).$$

Assumption 3(ii) implies

$$\sup_{\beta\in\mathcal{B}} \|S(\beta)^{-1}\|_2 = O(n^{-1}).$$

Since $M(\beta)$ is an orthogonal projection, $\|M(\beta)\|_2 = 1$. Therefore

$$\|D(\beta)\|_2 \le \|M(\beta)\|_2 \|A\|_2 \|S(\beta)^{-1}\|_2 \|R(\beta)\|_2 = O(1)$$

uniformly in β. Since $\operatorname{rank}(D(\beta)) \le K$,

$$\|D(\beta)\|_F \le \sqrt{K}\|D(\beta)\|_2 = O(\sqrt{n}),$$

where the last equality uses Assumption 3(iii).

The identity $\nabla_\beta M(\beta) = -(D(\beta) + D(\beta)')$ gives

$$\sup_{\beta \in \mathcal{B}} \|\nabla_\beta M(\beta)\|_2 = O(1).$$

Next, differentiating $D = MAS^{-1}R'$ and using $\nabla_\beta R = A$ and

$$\nabla_\beta S^{-1} = -S^{-1}(A'R + R'A)S^{-1},$$

yields

$$\nabla_\beta D = (\nabla_\beta M)AS^{-1}R' - MAS^{-1}(A'R + R'A)S^{-1}R' + MAS^{-1}A'.$$

The three terms have operator norms bounded by

$$\begin{aligned} &O(1)\, O(\sqrt{n})\, O(n^{-1})\, O(\sqrt{n}) = O(1), \\ &O(\sqrt{n})\, O(n^{-1})\, O(n)\, O(n^{-1})\, O(\sqrt{n}) = O(1), \\ &O(\sqrt{n})\, O(n^{-1})\, O(\sqrt{n}) = O(1), \end{aligned}$$

respectively, where $\|A'R + R'A\|_2 \leq 2\|A\|_2\|R\|_2 = O(n)$. Thus $\sup_{\beta \in \mathcal{B}} \|\nabla_\beta D(\beta)\|_2 = O(1)$.

For Λ, recall

$$\Lambda_{\ell\ell}(\beta) = \frac{\nabla_\beta M_{\ell\ell}(\beta)}{M_{\ell\ell}(\beta)}.$$

Assumption 3(i) bounds the denominator away from zero. Also

$$\nabla_\beta M_{\ell\ell}(\beta) = -2D_{\ell\ell}(\beta), \qquad |D_{\ell\ell}(\beta)| \leq \|D(\beta)\|_2,$$

so $\sup_{\beta \in \mathcal{B}} \|\Lambda(\beta)\|_\infty = O(1)$. Moreover,

$$\nabla_\beta \Lambda_{\ell\ell}(\beta) = \frac{\nabla_{\beta\beta} M_{\ell\ell}(\beta)}{M_{\ell\ell}(\beta)} - \frac{\{\nabla_\beta M_{\ell\ell}(\beta)\}^2}{M_{\ell\ell}(\beta)^2}.$$

Since $\nabla_{\beta\beta} M = -(\nabla_\beta D + (\nabla_\beta D)')$, the previous bounds and bounded leverage imply $\sup_{\beta \in \mathcal{B}} \|\nabla_\beta \Lambda(\beta)\|_\infty = O(1)$.

Finally, $U^{\mathrm{A}} = -(2D + M\Lambda)$. Hence

$$\|U^{\mathrm{A}}\|_2 \leq 2\|D\|_2 + \|M\|_2\|\Lambda\|_\infty = O(1),$$

and

$$\|\nabla_\beta U^{\mathrm{A}}\|_2 \leq 2\|\nabla_\beta D\|_2 + \|\nabla_\beta M\|_2\|\Lambda\|_\infty + \|M\|_2\|\nabla_\beta \Lambda\|_\infty = O(1).$$

For the Frobenius norm,

$$\|M\Lambda\|_F^2 = \operatorname{tr}(\Lambda M' M \Lambda) = \operatorname{tr}(\Lambda^2 M) \leq \|\Lambda\|_\infty^2 \operatorname{tr}(M) = O(n),$$

and therefore $\|U^{\mathrm{A}}\|_F \leq 2\|D\|_F + \|M\Lambda\|_F = O(\sqrt{n})$. The same product-rule calculations give

$$\sup_{\beta\in\mathcal{B}} \|\nabla_\beta D(\beta)\|_F = O(\sqrt{n}), \qquad \sup_{\beta\in\mathcal{B}} \|\nabla_\beta M(\beta)\|_F = O(\sqrt{n}).$$

Combining these bounds with $\|\Lambda\|_\infty = O(1)$, $\|\nabla_\beta \Lambda\|_\infty = O(1)$, and $\operatorname{tr}(M) = O(n)$ yields

$$\sup_{\beta\in\mathcal{B}} \|\nabla_\beta U^{\mathrm{A}}(\beta)\|_F = O(\sqrt{n}).$$

Differentiating the expression for $\nabla_\beta U^{\mathrm{A}}$ once more produces only finite sums of products of A, R, powers of S^{-1}, M, Λ, and their first derivatives. The preceding operator-norm bounds, together with bounded leverage, imply

$$\sup_{\beta\in\mathcal{B}} \|\nabla_{\beta\beta} U^{\mathrm{A}}(\beta)\|_2 = O(1).$$

□

Lemma 3 (Uniform convergence of the CF moment)**.** *Under Assumptions 1, 2, and 3,*

$$\sup_{\beta\in\mathcal{B}} n^{-1}|\hat{m}_n^{\mathrm{CF}}(\beta) - m_n^{\mathrm{CF}}(\beta)| \xrightarrow{p} 0.$$

Proof. Let $\mu = \mathbb{E}[y \mid X, A] = R(\beta_0)\delta$ and $\varepsilon = y - \mu$. Assumption 3(ii) implies

$$\max_{\ell \le n} |\mu_\ell| \le (C_x + |\beta_0| C_a)\, C_\delta, \qquad \|\mu\|_2^2 = O(n).$$

The matrix representation in Section 4 gives

$$\hat{m}_n^{\mathrm{CF}}(\beta) = y' U^{\mathrm{A}}(\beta) y, \qquad U^{\mathrm{A}}(\beta) = -(2D(\beta) + M(\beta)\Lambda(\beta)).$$

The diagonal of U^{A} is zero because the diagonal of $2D$ is $-\nabla_\beta M_{\ell\ell}$ and the diagonal of $M\Lambda$ is $\nabla_\beta M_{\ell\ell}$. Hence, using conditional independence,

$$m_n^{\mathrm{CF}}(\beta) = \mathbb{E}[y' U^{\mathrm{A}}(\beta) y \mid X, A] = \mu' U^{\mathrm{A}}(\beta) \mu.$$

Writing $U^{\mathrm{S}} = (U^{\mathrm{A}} + U^{\mathrm{A}\prime})/2$, we have

$$\hat{m}_n^{\mathrm{CF}}(\beta) - m_n^{\mathrm{CF}}(\beta) = 2\mu' U^{\mathrm{S}}(\beta)\varepsilon + \varepsilon' U^{\mathrm{A}}(\beta)\varepsilon.$$

Both terms have conditional mean zero. For the linear term,

$$\mathrm{Var}\Big(2\mu' U^{\mathrm{S}}(\beta)\varepsilon \mid X, A\Big) \le 4\|\mu\|_2^2 \|U^{\mathrm{S}}(\beta)\|_2^2 \max_{\ell \le n} \sigma_\ell^2 = O(n),$$

uniformly in fixed β, since $\|U^{\mathrm{S}}\|_2 \le \|U^{\mathrm{A}}\|_2 = O(1)$ and Assumption 1(iv) bounds $\max_\ell \sigma_\ell^2$. For the quadratic term, the diagonal-zero property, conditional independence, and bounded fourth moments imply the standard quadratic-form bound

$$\mathrm{Var}\Big(\varepsilon' U^{\mathrm{A}}(\beta)\varepsilon \mid X, A\Big) \le C\|U^{\mathrm{A}}(\beta)\|_F^2 = O(n).$$

Thus, for each fixed $\beta \in \mathcal{B}$,

$$n^{-1}|\hat{m}_n^{\mathrm{CF}}(\beta) - m_n^{\mathrm{CF}}(\beta)| \xrightarrow{p} 0.$$

We now upgrade pointwise convergence to uniform convergence. First,

$$\|y\|_2^2 \le 2\|\mu\|_2^2 + 2\|\varepsilon\|_2^2 = O_p(n),$$

because $\mathbb{E}[\|\varepsilon\|_2^2 \mid X, A] = \sum_\ell \sigma_\ell^2 = O(n)$. By the mean-value theorem and Lemma 2,

$$\begin{aligned} n^{-1}|\hat{m}_n^{\mathrm{CF}}(\beta) - \hat{m}_n^{\mathrm{CF}}(\beta')| &\le n^{-1}\|y\|_2^2 \sup_{\tilde{\beta}\in\mathcal{B}} \|\nabla_\beta U^{\mathrm{A}}(\tilde{\beta})\|_2 |\beta - \beta'| \\ &\le B_n|\beta - \beta'|, \end{aligned}$$

where $B_n = O_p(1)$. Similarly,

$$n^{-1}|m_n^{\mathrm{CF}}(\beta) - m_n^{\mathrm{CF}}(\beta')| \le n^{-1}\|\mu\|_2^2 \sup_{\tilde{\beta}\in\mathcal{B}} \|\nabla_\beta U^{\mathrm{A}}(\tilde{\beta})\|_2 |\beta - \beta'| \le C|\beta - \beta'|.$$

Let $\delta_\eta > 0$ be a mesh size to be chosen and let $\{\beta_j\}_{j=1}^{J_\eta}$ be a finite δ_η-net of the compact set $\mathcal{B}$. For any $\beta \in \mathcal{B}$, choose β_j with $|\beta - \beta_j| \le \delta_\eta$. Then

$$\begin{aligned} &n^{-1}|\hat{m}_n^{\mathrm{CF}}(\beta) - m_n^{\mathrm{CF}}(\beta)| \\ &\quad\le \max_{1\le j\le J_\eta} n^{-1}|\hat{m}_n^{\mathrm{CF}}(\beta_j) - m_n^{\mathrm{CF}}(\beta_j)| + (B_n + C)\delta_\eta. \end{aligned}$$

The maximum over the finite net is $o_p(1)$ by pointwise convergence. Since $B_n = O_p(1)$, choosing δ_η arbitrarily small gives the desired uniform convergence. □

The theorem now follows immediately from Assumption 3(iv), Lemma 3, and Lemma 1. □

C.2 Proof of Theorem 2

We prove the two claims in Theorem 2 in three steps. First, we establish the central limit theorem for the CF moment. Let $\varepsilon = y - \mu$. At β_0, Proposition 1 gives $m_n^{\mathrm{CF}}(\beta_0) = 0$, and the diagonal-zero property of U^{A} gives

$$\begin{aligned} \hat{m}_n^{\mathrm{CF}}(\beta_0) &= 2\mu' U^{\mathrm{S}}(\beta_0)\varepsilon + \varepsilon' U^{\mathrm{S}}(\beta_0)\varepsilon \\ &= \sum_{\ell=1}^{n} c_{\ell n}\varepsilon_\ell + \sum_{\ell<k} 2U_{\ell k}^{\mathrm{S}}(\beta_0)\varepsilon_\ell\varepsilon_k. \end{aligned}$$

This is a generalized quadratic form in the independent errors $\{\varepsilon_\ell\}$, consisting of a linear projection term and a degenerate quadratic term. Assumption 4(i) gives the required uniform moment bound. It remains to verify that no single observation

dominates. We first claim

$$\max_{\ell \le n} \sum_{k=1}^{n} |U^{\mathrm{S}}_{\ell k}(\beta_0)| = O(1).$$

To see this, let $r_\ell(\beta) = x_\ell + a_\ell \beta$. Bounded row ℓ_1 norms imply $\sup_{\ell, \beta \in \mathcal{B}} \|r_\ell(\beta)\|_2 = O(1)$. Since $\|S(\beta)^{-1}\|_2 = O(n^{-1})$,

$$\sum_{k=1}^{n} |P_{\ell k}(\beta)| = \sum_{k=1}^{n} |r_\ell(\beta)' S(\beta)^{-1} r_k(\beta)| \le \|S(\beta)^{-1} r_\ell(\beta)\|_2 \sum_{k=1}^{n} \|r_k(\beta)\|_2 = O(1),$$

uniformly in ℓ and β. Hence the row and column ℓ_1 norms of $M(\beta) = I - P(\beta)$ are uniformly bounded. It follows that

$$\max_{\ell \le n} \|e_\ell' M(\beta) A\|_1 \le \max_{\ell \le n} \sum_{j=1}^{n} |M_{\ell j}(\beta)| \|a_j\|_1 = O(1),$$

where we use $\max_j \|a_j\|_1 = O(1)$. Using this bound and $\|S(\beta)^{-1}\|_2 = O(n^{-1})$ gives uniformly bounded row and column ℓ_1 norms for

$$D(\beta) = M(\beta) A S(\beta)^{-1} R(\beta)'.$$

Moreover, $\|\Lambda(\beta)\|_\infty = O(1)$ by Lemma 2, so $M(\beta)\Lambda(\beta)$ also has uniformly bounded row and column ℓ_1 norms. Since $U^{\mathrm{A}} = -(2D + M\Lambda)$ and $U^{\mathrm{S}} = (U^{\mathrm{A}} + U^{\mathrm{A}\prime})/2$, the displayed claim follows. The same calculations, using the derivative bounds in Lemma 2, give uniformly bounded row and column ℓ_1 norms for $U^{\mathrm{A}}(\beta)$, $U^{\mathrm{S}}(\beta)$, $\nabla_\beta U^{\mathrm{A}}(\beta)$, and $\nabla_\beta U^{\mathrm{S}}(\beta)$ on $\mathcal{N}_0$. Let

$$W_{\ell k m}(\beta) = 2 U^{\mathrm{S}}_{\ell k}(\beta) U^{\mathrm{A}}_{\ell m}(\beta),$$

$$W^{(0)}_{\ell k m}(\beta) = W_{\ell k m}(\beta), \quad W^{(1)}_{\ell k m}(\beta) = \nabla_\beta W_{\ell k m}(\beta),$$
$$U^{(0)}_{\ell k}(\beta) = U^{\mathrm{A}}_{\ell k}(\beta), \quad U^{(1)}_{\ell k}(\beta) = \nabla_\beta U^{\mathrm{A}}_{\ell k}(\beta),$$

and

$$h_{\ell s}^{(-\ell km)}(\beta) = \mathbb{I}\{s \notin I_{\ell km}\} r_\ell(\beta)' S_{-I_{\ell km}}(\beta)^{-1} r_s(\beta),$$
$$I_{\ell km} = \{\ell, k, m\}, \qquad S_{-I}(\beta) = \sum_{j \notin I} r_j(\beta) r_j(\beta)'.$$

$$h_{\ell s}^{(-\ell km,0)}(\beta) = h_{\ell s}^{(-\ell km)}(\beta), \quad h_{\ell s}^{(-\ell km,1)}(\beta) = \nabla_\beta h_{\ell s}^{(-\ell km)}(\beta).$$

The row and column bounds for U^{A} and U^{S} imply, for $q = 0, 1$,

$$\sup_{\beta \in \mathcal{N}_0} \max_{\ell} \sum_{k,m} |W_{\ell km}^{(q)}(\beta)| + \sup_{\beta \in \mathcal{N}_0} \max_{k} \sum_{\ell,m} |W_{\ell km}^{(q)}(\beta)|$$
$$+ \sup_{\beta \in \mathcal{N}_0} \max_{m} \sum_{\ell,k} |W_{\ell km}^{(q)}(\beta)| = O(1).$$

We next verify the leave-out weight bounds. Let $R_I(\beta)$ be the matrix whose rows are $\{r_i(\beta)' : i \in I\}$. For a leave-out set $I = I_{\ell km}$, the Sherman–Morrison–Woodbury formula gives

$$S_{-I}(\beta)^{-1} = S(\beta)^{-1} + S(\beta)^{-1} R_I(\beta)' M_{II}(\beta)^{-1} R_I(\beta) S(\beta)^{-1},$$

where $M_{II}(\beta) = I_{|I|} - R_I(\beta) S(\beta)^{-1} R_I(\beta)'$ is the principal submatrix of $M(\beta)$ indexed by I. By the convention $D_{\ell kk}(\beta) = D_{\ell k}(\beta)$, $\det M_{I_{\ell km} I_{\ell km}}(\beta) = D_{\ell km}(\beta)$. Since $0 \preceq M_{II}(\beta) \preceq I_{|I|}$, Assumption 4(ii) implies $\sup_{\beta \in \mathcal{N}_0, \ell, k, m} \| M_{I_{\ell km} I_{\ell km}}(\beta)^{-1} \|_2 = O(1)$. Combining this with Assumption 3(ii) and bounded row ℓ_1 norms of x_ℓ and a_ℓ yields

$$\sup_{\beta \in \mathcal{N}_0, \ell, k, m} \| S_{-I_{\ell km}}(\beta)^{-1} \|_2 = O(n^{-1}).$$

Moreover, $\sup_{\beta \in \mathcal{N}_0, \ell, k, m} \| \nabla_\beta S_{-I_{\ell km}}(\beta) \|_2 = O(n)$, so

$$\nabla_\beta S_{-I}(\beta)^{-1} = -S_{-I}(\beta)^{-1} \{\nabla_\beta S_{-I}(\beta)\} S_{-I}(\beta)^{-1}$$

has operator norm $O(n^{-1})$ uniformly in β and I. Therefore

$$\sup_{\beta \in \mathcal{N}_0} \max_{\ell, k, m, s} |h_{\ell s}^{(-\ell km)}(\beta)| = O(n^{-1}), \qquad \sup_{\beta \in \mathcal{N}_0} \max_{\ell, k, m} \sum_{s} |h_{\ell s}^{(-\ell km)}(\beta)| = O(1),$$

and the same bounds with $h_{\ell s}^{(-\ell km,1)}(\beta)$ in place of $h_{\ell s}^{(-\ell km,0)}(\beta)$.

These primitive bounds imply the overlap control needed for the leave-three-out variance estimator. For each observation i, define

$$\mathcal{K}_{q,i}(\beta) = \sum_{\ell,k,m:\, i\in\{\ell,k,m\}} |W_{\ell km}^{(q)}(\beta)| + \sum_{\ell,k,m,s:\, i\in\{\ell,k,m,s\}} |W_{\ell km}^{(q)}(\beta) h_{\ell s}^{(-\ell km,0)}(\beta)| + \sum_{\ell,k,m,s:\, i\in\{\ell,k,m,s\}} |W_{\ell km}^{(0)}(\beta) h_{\ell s}^{(-\ell km,q)}(\beta)| + \sum_{j=1}^{n} \left\{ |U_{ij}^{(q)}(\beta)| + |U_{ji}^{(q)}(\beta)| \right\}.$$

The first and last terms are $O(1)$ uniformly in i, q, and $\beta \in \mathcal{N}_0$ by the slice bounds for W and the row and column bounds for U^{A}. For $q, r \in \{0,1\}$,

$$\sup_{\beta\in\mathcal{N}_0} \max_i \sum_{\ell,k,m,s:\, i\in\{\ell,k,m,s\}} |W_{\ell km}^{(q)}(\beta) h_{\ell s}^{(-\ell km,r)}(\beta)| \le C + \sup_{\beta\in\mathcal{N}_0} \left(\sum_{\ell,k,m} |W_{\ell km}^{(q)}(\beta)| \right) \max_{\ell,k,m,s} |h_{\ell s}^{(-\ell km,r)}(\beta)| = O(1),$$

because the total $W^{(q)}$ mass is $O(n)$ and $\max_{\ell,k,m,s} |h_{\ell s}^{(-\ell km,r)}(\beta)| = O(n^{-1})$. Therefore

$$\max_i \mathcal{K}_{0,i}(\beta_0) = O(1), \qquad \sup_{\beta\in\mathcal{N}_0} \max_i \mathcal{K}_{1,i}(\beta) = O(1).$$

The claim and $\max_\ell |\mu_\ell| = O(1)$ imply

$$\max_{\ell\le n} c_{\ell n}^2 \sigma_\ell^2 = O(1),$$

while Assumption 4(i) implies $\max_\ell \sigma_\ell^2 = O(1)$ and therefore

$$\max_{\ell\le n} \sum_{k\neq\ell} \omega_{\min\{\ell,k\},\max\{\ell,k\},n}^2 = O(1).$$

Since $\liminf_n n^{-1} V_n(\beta_0) > 0$, both maxima are $o(V_n(\beta_0))$. Thus the Lindeberg/no-dominant-component condition for the linear-plus-degenerate-quadratic form holds. The central-limit theorem for generalized quadratic forms in independent variables

(de Jong, 1987) therefore implies

$$\frac{\hat{m}_n^{\mathrm{CF}}(\beta_0)}{\sqrt{\mathrm{Var}(\hat{m}_n^{\mathrm{CF}}(\beta_0) \mid X, A)}} \xrightarrow{d} N(0,1).$$

The denominator is $V_n(\beta_0)$ by definition. Hence

$$\frac{\hat{m}_n^{\mathrm{CF}}(\beta_0)}{\sqrt{V_n(\beta_0)}} \xrightarrow{d} N(0,1). \tag{18}$$

Second, we prove consistency of the leave-three-out variance estimator. Let $r_\ell = x_\ell + a_\ell\beta_0$ and $I_{\ell km} = \{\ell, k, m\}$. At the true value,

$$\hat{\delta}_{(\ell km)}(\beta_0) = \delta + S_{-I_{\ell km}}(\beta_0)^{-1} \sum_{s \notin I_{\ell km}} r_s \varepsilon_s,$$

and therefore

$$\hat{\sigma}^2_{\ell,-km}(\beta_0) = y_\ell \left(\varepsilon_\ell - \sum_{s=1}^{n} h_{\ell s}^{(-\ell km)}(\beta_0)\varepsilon_s \right).$$

Substituting this expression into (10), and using $\hat{m}_n^{\mathrm{CF}}(\beta_0) = y' U^{\mathrm{A}}(\beta_0) y$, shows that $\hat{V}_n(\beta_0) - V_n(\beta_0)$ is a centered fourth-degree polynomial in the independent errors. Its deterministic weights are finite sums of products formed from

$$W_{\ell km}(\beta_0), \qquad W_{\ell km}(\beta_0) h_{\ell s}^{(-\ell km)}(\beta_0), \qquad U_{ij}^{\mathrm{A}}(\beta_0).$$

If two monomials in this polynomial involve disjoint sets of error indices, their covariance is zero. If they overlap, Assumption 4(i) bounds the covariance by a constant times the product of the corresponding absolute weights. Summing these possible overlaps gives

$$\mathrm{Var}\Big(\hat{V}_n(\beta_0) \mid X, A\Big) \le C \sum_{i=1}^{n} \mathcal{K}_{0,i}(\beta_0)^2.$$

The displayed overlap bound therefore gives

$$\mathrm{Var}\Big(\hat{V}_n(\beta_0) \mid X, A\Big) = o(n^2).$$

Since $\liminf_n n^{-1}V_n(\beta_0) > 0$, this implies

$$\mathrm{Var}\Big(\hat{V}_n(\beta_0) \mid X, A\Big) = o\big(V_n(\beta_0)^2\big)\,.$$

By Proposition 2,

$$\mathbb{E}[\hat{V}_n(\beta_0) \mid X, A] = V_n(\beta_0).$$

Therefore, for any $\varepsilon > 0$, Chebyshev's inequality gives

$$\mathbb{P}\left(\left| \frac{\hat{V}_n(\beta_0)}{V_n(\beta_0)} - 1 \right| > \varepsilon \,\middle|\, X, A \right) \leq \frac{\mathrm{Var}(\hat{V}_n(\beta_0) \mid X, A)}{\varepsilon^2 V_n(\beta_0)^2} = o(1),$$

Hence

$$\hat{V}_n(\beta_0)/V_n(\beta_0) \xrightarrow{p} 1,$$

By Theorem 1, $\hat{\beta}^{\mathrm{CF}} \xrightarrow{p} \beta_0$. Hence, with probability approaching one, $\hat{\beta}^{\mathrm{CF}} \in \mathcal{N}_0$. The derivative of the centered variance-estimator process is again a finite polynomial in the errors. Its non-random terms are $O(n)$ by the uniform slice bounds for W, h, U^{A}, and their first derivatives established above. For the centered terms, the same overlap argument, now using $\sup_{\beta \in \mathcal{N}_0} \mathcal{K}_{1,i}(\beta) = O(1)$, gives

$$\mathbb{E}\left[\sup_{\beta \in \mathcal{N}_0} \left\{ n^{-1} \left| \nabla_\beta \left(\hat{T}_n(\beta) - \{\hat{m}_n^{\mathrm{CF}}(\beta)\}^2 \right) \right| \right\}^2 \mid X, A \right] = O(1).$$

Therefore

$$n^{-1}|\hat{V}_n(\hat{\beta}^{\mathrm{CF}}) - \hat{V}_n(\beta_0)| \leq \sup_{\beta \in \mathcal{N}_0} n^{-1} \left| \nabla_\beta \left(\hat{T}_n(\beta) - \{\hat{m}_n^{\mathrm{CF}}(\beta)\}^2 \right) \right| |\hat{\beta}^{\mathrm{CF}} - \beta_0| = o_p(1).$$

Since $\liminf_n n^{-1}V_n(\beta_0) > 0$, this implies

$$\hat{V}_n(\hat{\beta}^{\mathrm{CF}})/V_n(\beta_0) \xrightarrow{p} 1.$$

This proves part (i).

Third, we prove the studentized normality. By Theorem 1, $\hat{\beta}^{\mathrm{CF}} \xrightarrow{p} \beta_0$. Since β_0 is in the interior of $\mathcal{B}$, with probability approaching one the line segment between $\hat{\beta}^{\mathrm{CF}}$

and β_0 lies in $\mathcal{N}_0$. A mean-value expansion of the sample moment around β_0 gives

$$0 = \hat{m}_n^{\mathrm{CF}}(\beta_0) + \nabla_\beta \hat{m}_n^{\mathrm{CF}}(\tilde{\beta})\big(\hat{\beta}^{\mathrm{CF}} - \beta_0\big)$$

for some $\tilde{\beta}$ between $\hat{\beta}^{\mathrm{CF}}$ and β_0. Rearranging,

$$\hat{\beta}^{\mathrm{CF}} - \beta_0 = -\frac{\hat{m}_n^{\mathrm{CF}}(\beta_0)}{\nabla_\beta \hat{m}_n^{\mathrm{CF}}(\tilde{\beta})}. \tag{19}$$

We first analyze the derivative terms. Differentiating the quadratic-form representation gives

$$\nabla_\beta \hat{m}_n^{\mathrm{CF}}(\beta) = y' \nabla_\beta U^{\mathrm{A}}(\beta) y.$$

Because the diagonal of $U^{\mathrm{A}}(\beta)$ is identically zero in β, the diagonal of $\nabla_\beta U^{\mathrm{A}}(\beta)$ is also zero. Hence

$$\nabla_\beta m_n^{\mathrm{CF}}(\beta) = \mathbb{E}[y' \nabla_\beta U^{\mathrm{A}}(\beta) y \mid X, A] = \mu' \nabla_\beta U^{\mathrm{A}}(\beta) \mu.$$

The proof of Lemma 3, applied with $\nabla_\beta U^{\mathrm{A}}$ in place of U^{A} and using the bounds

$$\sup_{\beta \in \mathcal{B}} \|\nabla_\beta U^{\mathrm{A}}(\beta)\|_2 = O(1), \quad \sup_{\beta \in \mathcal{B}} \|\nabla_\beta U^{\mathrm{A}}(\beta)\|_F = O(\sqrt{n}), \quad \sup_{\beta \in \mathcal{B}} \|\nabla_{\beta\beta} U^{\mathrm{A}}(\beta)\|_2 = O(1),$$

from Lemma 2, gives

$$\sup_{\beta \in \mathcal{B}} n^{-1} |\nabla_\beta \hat{m}_n^{\mathrm{CF}}(\beta) - \nabla_\beta m_n^{\mathrm{CF}}(\beta)| \xrightarrow{p} 0. \tag{20}$$

Moreover, for any two sequences $\beta_n, \beta_n' \in \mathcal{N}_0$ satisfying $|\beta_n - \beta_n'| = o_p(1)$,

$$n^{-1} |\nabla_\beta m_n^{\mathrm{CF}}(\beta_n) - \nabla_\beta m_n^{\mathrm{CF}}(\beta_n')| \le n^{-1} \|\mu\|_2^2 \sup_{\beta \in \mathcal{B}} \|\nabla_{\beta\beta} U^{\mathrm{A}}(\beta)\|_2 |\beta_n - \beta_n'| = o_p(1),$$

because $\|\mu\|_2^2 = O(n)$. Combining this display with (20) yields

$$\frac{\nabla_\beta \hat{m}_n^{\mathrm{CF}}(\hat{\beta}^{\mathrm{CF}})}{\nabla_\beta \hat{m}_n^{\mathrm{CF}}(\tilde{\beta})} \xrightarrow{p} 1.$$

To see this, the numerator of the difference between this ratio and one is $o_p(n)$ by the preceding two displays, while the local slope condition in the theorem and (20)

imply that

$$\inf_{\beta \in \mathcal{N}_0} n^{-1} |\nabla_\beta \hat{m}_n^{\mathrm{CF}}(\beta)|$$

is bounded away from zero with probability approaching one.

Using (19), we can write the studentized statistic as

$$\begin{aligned}\frac{\nabla_\beta \hat{m}_n^{\mathrm{CF}}(\hat{\beta}^{\mathrm{CF}})}{\sqrt{\hat{V}_n(\hat{\beta}^{\mathrm{CF}})}}\big(\hat{\beta}^{\mathrm{CF}} - \beta_0\big) &= -\frac{\hat{m}_n^{\mathrm{CF}}(\beta_0)}{\sqrt{V_n(\beta_0)}}\sqrt{\frac{V_n(\beta_0)}{\hat{V}_n(\hat{\beta}^{\mathrm{CF}})}}\frac{\nabla_\beta \hat{m}_n^{\mathrm{CF}}(\hat{\beta}^{\mathrm{CF}})}{\nabla_\beta \hat{m}_n^{\mathrm{CF}}(\tilde{\beta})} \\ &\xrightarrow{d} N(0,1),\end{aligned}$$

where we use the central limit theorem in (18), the plug-in variance consistency in part (i), and the derivative ratio above. The minus sign does not affect the limit because the standard normal distribution is symmetric. □

C.3 Proof of Proposition 1

Observe that

$$\mathbb{E}[\hat{m}_n(\beta_0)|X, A] = m_n(\beta)_0 = \sum_{l=1}^{n} \nabla_\beta M_{ll}(\beta_0)\sigma_l^2.$$

Also, if $\min_{l \le n} M_{ll}(\beta_0) > 0$, each $\hat{\sigma}_l^2(\beta_0)$ is well-defined and unbiased for σ_l^2:

$$\begin{aligned}\mathbb{E}[\hat{\sigma}_l^2(\beta_0)|X, A] &= \mathbb{E}[(\mu_l + \varepsilon_l)(\mu_l + \varepsilon_l - (x_l + a_l\beta_0)'\hat{\delta}_{(l)}^{\mathrm{LS}}(\beta_0))|X, A] \\ &= \mathbb{E}[(\mu_l + \varepsilon_l)(\varepsilon_l + (x_l + a_l\beta_0)'(\delta - \hat{\delta}_{(l)}^{\mathrm{LS}}(\beta_0)))|X, A] \\ &= \sigma_l^2\end{aligned}$$

where the last equality uses the independence of ε_l from $\hat{\delta}_{(l)}^{\mathrm{LS}}(\beta_0)$ and the fact that $\hat{\delta}_{(l)}^{\mathrm{LS}}(\beta_0)$ is an unbiased estimator of δ under the leave-one-out procedure:

$$\mathbb{E}[\hat{\delta}_{(l)}^{\mathrm{LS}}(\beta_0)|X, A] = \left(\sum_{j \neq l}(x_j + a_j\beta_0)(x_j + a_j\beta_0)'\right)^{-1} \sum_{j \neq l}(x_j + a_j\beta_0)(x_j + a_j\beta_0)'\delta = \delta.$$

Therefore,

$$\mathbb{E}[\hat{m}_n^{\mathrm{CF}}(\beta_0)] = \mathbb{E}[\hat{m}_n(\beta_0)|X, A] - \sum_{l=1}^{n} \nabla_\beta M_{ll}(\beta_0)\mathbb{E}[\hat{\sigma}_l^2(\beta_0)|X, A] = 0.$$

This completes the proof of Proposition 1. □

C.4 Proof of Proposition 2

Observe that

$$\begin{aligned}
\mathbb{E}[\hat{\sigma}^2_{l,-km}(\beta_0)|X, A] &= \mathbb{E}[(\mu_l + \varepsilon_l)(\mu_l + \varepsilon_l - (x_l + a_l\beta_0)'\hat{\delta}_{(lkm)}(\beta_0))|X, A] \\
&= \mathbb{E}[(\mu_l + \varepsilon_l)(\varepsilon_l + (x_l + a_l\beta_0)'(\delta - \hat{\delta}_{(lkm)}(\beta_0)))|X, A] \\
&= \sigma_l^2,
\end{aligned}$$

where the last equality uses the independence of ε_l from $\hat{\delta}_{(lkm)}(\beta_0)$ and the fact that $\hat{\delta}_{(lkm)}(\beta_0)$ is an unbiased estimator of δ under the leave-three-out procedure:

$$\begin{aligned}
&\mathbb{E}[\hat{\delta}_{(lkm)}(\beta_0)|X, A] \\
&= \left(\sum_{j\notin\{l,k,m\}} (x_j + a_j\beta_0)(x_j + a_j\beta_0)'\right)^{-1} \sum_{j\notin\{l,k,m\}} (x_j + a_j\beta_0)(x_j + a_j\beta_0)'\delta \\
&= \delta.
\end{aligned}$$

Finally, note that

$$\mathbb{E}[\hat{m}_n^{\mathrm{CF}}(\beta_0)|X, A] = \mathrm{Var}(\hat{m}_n(\beta_0)|X, A) = V_n(\beta_0),$$

where the first equality uses Proposition 1 and the second equality uses the definition of $V_n(\beta_0)$ in (10). Therefore,

$$\begin{aligned}
&\mathbb{E}[\hat{V}_n(\beta_0)|X, A] \\
&= 2\mathbb{E}\left[\sum_{l=1}^{n}\sum_{k\neq l}\sum_{m\neq l} U_{ik}^S(\beta_0)U_{lm}^A(\beta_0)y_k y_m \hat{\sigma}^2_{l,-km}(\beta_0)|X, A\right] - \mathbb{E}[(\hat{m}_n^{\mathrm{CF}}(\beta_0))^2|X, A] \\
&= 2\mathbb{E}\left[\sum_{l=1}^{n}\sum_{k\neq l}\sum_{m\neq l} U_{ik}^S(\beta_0)U_{lm}^A(\beta_0)y_k y_m \sigma_l^2|X, A\right] - V_n(\beta_0) \\
&= 0,
\end{aligned}$$

where the second equality uses the independence of $\hat{\sigma}^2_{l,-km}(\beta_0)$ from y_k and y_m for $k, m \neq l$, and the last equality uses the expressions of $V_n(\beta_0)$ in (10). This completes the proof of Proposition 2. □

C.5 Proof of Proposition 3

Lemma 4 (Concentration of the conservative variance estimator)**.** *Suppose Assumptions 1, 2, and 3 hold, Assumption 4(i) holds, Assumption 5 replaces Assumption 4(ii), and* $\liminf_{n\to\infty} n^{-1}V_n(\beta_0) > 0$*. Then for every* $\eta > 0$*,*

$$\mathbb{P}\left(\sup_{\beta\in\mathcal{N}_0}\frac{|\hat{V}_n^{\mathrm{mod}}(\beta) - \bar{V}_n^{\mathrm{mod}}(\beta)|}{V_n(\beta_0)} > \eta \,\middle|\, X, A\right) \to 0.$$

Proof. The proof follows the overlap argument used for $\hat{V}_n$ in the proof of Theorem 2. Assumption 5, together with Assumption 3(i), gives the same row and column bounds for the leave-two-out and leave-three-out projection weights that are actually used in (21). The replacement y_ℓ^2 introduces no leave-out projection weights, and the trimming indicators $G_{\ell,-km}$ are bounded by one. Hence the deterministic absolute-weight sums entering the overlap calculation are bounded by the same $\mathcal{K}_{q,i}$ envelopes constructed above, up to constants.

Using bounded eighth moments, covariance terms whose error indices overlap are bounded by a constant times the product of their absolute weights. Terms with disjoint relevant indices have zero covariance after conditioning on the variables entering the corresponding trimming indicators; the remaining covariance terms are covered

by the same overlap envelope because each trimming indicator is attached to a single ℓ-slice. Therefore

$$\mathrm{Var}\Big(\hat{V}_n^{\mathrm{mod}}(\beta) - \bar{V}_n^{\mathrm{mod}}(\beta) \mid X, A\Big) \le C \sum_{i=1}^{n} \mathcal{K}_{0,i}(\beta)^2 = o(n^2)$$

uniformly on $\mathcal{N}_0$. Since $\liminf_n n^{-1} V_n(\beta_0) > 0$, Chebyshev's inequality gives pointwise convergence after normalization by $V_n(\beta_0)$. The derivative version of the same overlap bound gives stochastic equicontinuity on $\mathcal{N}_0$, so the pointwise convergence upgrades to the displayed uniform convergence. □

Lemma 5 (Conservative centering)**.** *For fixed β, write $\bar{V}_n^{\mathrm{mod}}(\beta) = \mathbb{E}[\hat{V}_n^{\mathrm{mod}}(\beta) \mid X, A]$ and $\Delta_n^{\mathrm{mod}}(\beta) = \bar{V}_n^{\mathrm{mod}}(\beta) - V_n(\beta)$. Under the conditions of Lemma 4,*

$$\Delta_n^{\mathrm{mod}}(\beta_0) \ge 0.$$

Moreover, for every $\eta > 0$,

$$\mathbb{P}\left(\frac{\bar{V}_n^{\mathrm{mod}}(\hat{\beta}^{\mathrm{CF}})}{V_n(\beta_0)} < 1 - \eta \,\middle|\, X, A \right) \to 0.$$

Proof. At β_0, the first two lines of (21) give unbiased replacements for the corresponding terms in (8): the original leave-three-out estimator is unbiased when $D_{\ell km}(\beta_0) > 0$, and the leave-two-out replacement is unbiased when the leave-three-out failure is not caused by ℓ.

It remains to consider the terms for which $\bar{\sigma}^2_{\ell,-km}(\beta_0) = y_\ell^2$. For such terms, y_ℓ^2 is independent of the aggregate weight

$$B_\ell(\beta_0) = \sum_{k \ne \ell} \sum_{m \ne \ell} W_{\ell km}(\beta_0) \mathbb{I}\big\{ \bar{\sigma}^2_{\ell,-km}(\beta_0) = y_\ell^2 \big\},$$

conditional on X, A and on $\{y_j : j \ne \ell\}$. If $B_\ell(\beta_0) \ge 0$, then $G_{\ell,-km}(\beta_0) = 1$ for the biased terms attached to ℓ, and their conditional excess relative to using σ_ℓ^2 is

$$B_\ell(\beta_0) \mathbb{E}[y_\ell^2 - \sigma_\ell^2 \mid X, A, \{y_j : j \ne \ell\}] = B_\ell(\beta_0) \mu_\ell^2 \ge 0.$$

If $B_\ell(\beta_0) < 0$, then $G_{\ell,-km}(\beta_0) = 0$ for these biased terms. Omitting them removes the target contribution $B_\ell(\beta_0)\sigma_\ell^2$, which is nonpositive, and therefore again weakly

increases the expectation of the variance estimator. Summing over ℓ gives $\Delta_n^{\mathrm{mod}}(\beta_0) \geq 0$.

For the plug-in statement, Theorem 1 gives $\hat{\beta}^{\mathrm{CF}} \xrightarrow{p} \beta_0$. The determinant condition in Assumption 5 makes the leave-out replacement rule locally stable except at bounded trimming boundaries, and the same derivative bounds used in the proof of Theorem 2 give

$$\frac{\bar{V}_n^{\mathrm{mod}}(\hat{\beta}^{\mathrm{CF}}) - \bar{V}_n^{\mathrm{mod}}(\beta_0)}{V_n(\beta_0)} \xrightarrow{p} 0.$$

Since $\bar{V}_n^{\mathrm{mod}}(\beta_0) \geq V_n(\beta_0)$, the displayed one-sided claim follows. □

Because $\bar{V}_n^{\mathrm{mod}}(\beta_0) \geq V_n(\beta_0)$ by Lemma 5,

$$\left(1 - \frac{\hat{V}_n^{\mathrm{mod}}(\beta_0)}{V_n(\beta_0)}\right)_+ \leq \frac{|\hat{V}_n^{\mathrm{mod}}(\beta_0) - \bar{V}_n^{\mathrm{mod}}(\beta_0)|}{V_n(\beta_0)}.$$

The first claim therefore follows from Lemma 4.

For the plug-in statement, use the deterministic inequality

$$\left(1 - \frac{\hat{V}_n^{\mathrm{mod}}(\hat{\beta}^{\mathrm{CF}})}{V_n(\beta_0)}\right)_+ \leq \left(1 - \frac{\bar{V}_n^{\mathrm{mod}}(\hat{\beta}^{\mathrm{CF}})}{V_n(\beta_0)}\right)_+ + \frac{|\hat{V}_n^{\mathrm{mod}}(\hat{\beta}^{\mathrm{CF}}) - \bar{V}_n^{\mathrm{mod}}(\hat{\beta}^{\mathrm{CF}})|}{V_n(\beta_0)}.$$

Lemma 5 makes the first term on the right negligible in probability. Theorem 1 implies $\hat{\beta}^{\mathrm{CF}} \in \mathcal{N}_0$ with probability approaching one, so Lemma 4 makes the second term negligible in probability. □

Appendix D When the leave-three-out variance estimator fails

The full-rank requirement in Assumption 4(ii) can fail in fixed-effect designs with small cells, because deleting three observations may remove the last identifying observation for one or more nuisance parameters. Following the logic of Section 4 of Anatolyev and Sølvsten (2020), the variance estimator can be modified so that the unbiased leave-three-out estimator is used whenever it exists, while failures are han-

dled by replacements that are either still unbiased for the relevant product or upward biased.

To state the modification, define the leave-two and leave-three determinants for distinct observations:

$$D_{\ell k}(\beta) = \begin{vmatrix} M_{\ell\ell}(\beta) & M_{\ell k}(\beta) \\ M_{\ell k}(\beta) & M_{kk}(\beta) \end{vmatrix},$$
$$D_{\ell km}(\beta) = \begin{vmatrix} M_{\ell\ell}(\beta) & M_{\ell k}(\beta) & M_{\ell m}(\beta) \\ M_{\ell k}(\beta) & M_{kk}(\beta) & M_{km}(\beta) \\ M_{\ell m}(\beta) & M_{km}(\beta) & M_{mm}(\beta) \end{vmatrix}.$$

By the Sherman-Morrison-Woodbury formula, $D_{\ell k}(\beta) > 0$ and $D_{\ell km}(\beta) > 0$ characterize the existence of the corresponding leave-two-out and leave-three-out least squares estimators, given full rank of the full-sample design. We use the conventions $D_{\ell kk}(\beta) = D_{\ell k}(\beta)$ and $D_{kk}(\beta) = 0$.

Define the replacement for the leave-three-out variance estimator by

$$\bar{\sigma}^2_{\ell,-km}(\beta) = \begin{cases} \hat{\sigma}^2_{\ell,-km}(\beta), & \text{if } D_{\ell km}(\beta) > 0, \\ \hat{\sigma}^2_{\ell,-k}(\beta), & \text{if } D_{km}(\beta) = 0 \text{ and } D_{\ell k}(\beta) D_{\ell m}(\beta) > 0, \\ y_\ell^2, & \text{otherwise.} \end{cases} \tag{21}$$

The first line is the original leave-three-out estimator. The second line applies when the leave-three-out failure is not caused by observation ℓ: in that case, the leave-two-out estimator for ℓ is independent of y_k and y_m, so the term $y_k y_m \hat{\sigma}^2_{\ell,-k}(\beta_0)$ remains unbiased for the component of (8). The final line applies when the failure is caused by ℓ; it replaces σ_ℓ^2 by y_ℓ^2, whose conditional expectation is $\sigma_\ell^2 + \mu_\ell^2$ at the truth, where $\mu_\ell = (x_\ell + a_\ell \beta_0)'\delta$.

Because the last line of (21) is upward biased only before it is multiplied by the possibly signed weight in (10), biased replacements with negative aggregate weight should be removed to ensure the conservativeness of the resulting variance estimator. To this end, define the term-specific weight

$$W_{\ell km}(\beta) = 2 U^{\mathrm{S}}_{\ell k}(\beta) U^{\mathrm{A}}_{\ell m}(\beta) y_k y_m$$

and the aggregate weight on biased replacements for observation ℓ as

$$B_\ell(\beta) = \sum_{k\neq\ell}\sum_{m\neq\ell} W_{\ell km}(\beta)\mathbb{I}\{\bar{\sigma}^2_{\ell,-km}(\beta) = y_\ell^2\},$$

and set

$$G_{\ell,-km}(\beta) = \begin{cases} 0, & \text{if } \bar{\sigma}^2_{\ell,-km}(\beta) = y_\ell^2 \text{ and } B_\ell(\beta) < 0, \\ 1, & \text{otherwise.} \end{cases}$$

The conservative variance estimator is then

$$\hat{V}_n^{\text{mod}}(\beta) = \sum_{\ell=1}^{n}\sum_{k\neq\ell}\sum_{m\neq\ell} W_{\ell km}(\beta)G_{\ell,-km}(\beta)\bar{\sigma}^2_{\ell,-km}(\beta) - \left(\hat{m}_n^{\text{CF}}(\beta)\right)^2. \tag{22}$$

When all leave-three-out estimators exist, $\bar{\sigma}^2_{\ell,-km} = \hat{\sigma}^2_{\ell,-km}$ and $G_{\ell,-km} = 1$, so (22) reduces to the original estimator (10). When some leave-three-out estimators fail, the construction keeps all unbiased replacements and retains upward-biased replacements only when their aggregate contribution is nonnegative.

To state a formal conservative consistency result, define $\bar{V}_n^{\text{mod}}(\beta) = \mathbb{E}[\hat{V}_n^{\text{mod}}(\beta) \mid X, A]$ and $\Delta_n^{\text{mod}}(\beta) = \bar{V}_n^{\text{mod}}(\beta) - V_n(\beta)$ for fixed β. The following condition replaces Assumption 4(ii).

Assumption 5 (Regularity with leave-out failures)**.** *Conditional on (X, A), the nonzero leave-out determinants are bounded away from zero: there exist constants $C_D < \infty$ and $N_D < \infty$ such that, for all $n \geq N_D$,*

$$\sup_{\beta\in\mathcal{N}_0}\left\{\max_{\ell,k:\, D_{\ell k}(\beta)>0} D_{\ell k}(\beta)^{-1} + \max_{\ell,k,m:\, D_{\ell km}(\beta)>0} D_{\ell km}(\beta)^{-1}\right\} \leq C_D,$$

where maxima over empty sets are set equal to zero.

Assumption 5 is the analog of Assumption 4(ii): together with the bounded-leverage condition in Assumption 3(i), it permits $D_{\ell k}(\beta)$ and $D_{\ell km}(\beta)$ to be zero, but prevents the nonzero leave-out denominators that are actually used from becoming arbitrarily small.

The following proposition shows that the modified variance estimator is asymptotically conservative:

***Proposition* 3** (Conservative variance consistency when leave-three-out may fail)**.** *Suppose Assumptions 1, 2, and 3 hold, Assumption 4(i) holds, and Assumption 5 replaces Assumption 4(ii). If* $\liminf_{n\to\infty} n^{-1}V_n(\beta_0) > 0$*, then, for every* $\eta > 0$*, the positive part of the shortfall from the target variance is asymptotically negligible:*

$$\left(1 - \frac{\hat{V}_n^{\text{mod}}(\beta_0)}{V_n(\beta_0)}\right)_+ \to_p 0,$$
$$\left(1 - \frac{\hat{V}_n^{\text{mod}}(\hat{\beta}^{\text{CF}})}{V_n(\beta_0)}\right)_+ \to_p 0$$

where $(t)_+ = \max\{t, 0\}$.

Proposition 3 shows tests and confidence intervals based on

$$\frac{\nabla_\beta \hat{m}_n^{\text{CF}}(\hat{\beta}^{\text{CF}})}{\sqrt{\hat{V}_n^{\text{mod}}(\hat{\beta}^{\text{CF}})}}(\hat{\beta}^{\text{CF}} - \beta_0)$$

are asymptotically valid, albeit potentially conservative. This follows because by Theorem 2, this statistic is asymptotically normal when the denominator is replaced by $\sqrt{V_n(\beta_0)}$, and Proposition 3 implies that the denominator $\sqrt{\hat{V}_n^{\text{mod}}(\hat{\beta}^{\text{CF}})}$ is asymptotically no smaller than $\sqrt{V_n(\beta_0)}$ with probability approaching one.[12]

Appendix E Endogenous peer effects: empirical extension

The empirical specification in the main text considers only contextual peer effects — influence operating through peer characteristics (here, peer ability $\bar{\alpha}_{(i)j}$). A natural extension also allows for endogenous peer effects: influence operating through peer outcomes. Bramoullé et al. (2009) establishes that this richer model is point-identified when peers interact through a network and the network is not perfectly transitive. The corresponding regression specification, building on (1), is

$$y_{ij} = \alpha_i + \bar{\alpha}_{(i)j} \cdot \beta + \bar{y}_{(i)j} \cdot \lambda + \psi_j + \varepsilon_{ij}, \tag{23}$$

[12] The asymptotic normality of $\nabla_\beta \hat{m}_n^{\text{CF}}(\hat{\beta}^{\text{CF}})/\sqrt{V_n(\beta_0)}(\hat{\beta}^{\text{CF}} - \beta_0)$ does not depend on the existence of leave-three-out estimators, so it continues to hold under the conditions of Proposition 3.

where $\bar{y}_{(i)j}$ is the average grade in the peer group, excluding student i. Note that $\bar{y}_{(i)j}$ is not observed by the student at the time of choosing effort (although it is observed by us ex post), so it captures the average peer outcome that the peer group is exposed to as a whole.

We estimate (23) using a natural extension of the cross-fit moment to the joint parameter (β, λ), and report the multiplier-effect summary $(\hat{\beta}+\hat{\lambda})/(1-\hat{\lambda})$ with delta-method standard errors. Results appear in Appendix Table 1 alongside the main-text NLLS and CF baseline estimates for ease of comparison. We caution that the formal asymptotic theory for the joint estimator extends the arguments of Section 3–Section 4 but requires additional structure beyond Assumptions 1–3 — in particular, a quantitative identification condition on the network suitable to Bramoullé et al. (2009)-type arguments. Such an extension lies outside the scope of the present paper, and we report the estimates here as suggestive empirical evidence.

For Spring 2019 (pre-COVID), the multiplier effect $(\hat{\beta} + \hat{\lambda})/(1 - \hat{\lambda})$ is estimated at 0.110 with a delta-method standard error of 0.034, statistically distinguishable from zero. The endogenous coefficient $\hat{\lambda} = 0.075$ is also statistically significant. The corresponding one-standard-deviation effect is 0.021, about 35% smaller than under the contextual-only specification, suggesting that the endogenous channel absorbs part of what would otherwise be attributed to the contextual channel.

For Spring 2020 (COVID-online), the multiplier effect is estimated at -0.034 with a delta-method standard error of 0.033, statistically indistinguishable from zero. The conclusion of the main text — that the COVID-online semester eliminates the classroom peer effect — is robust to allowing endogenous peer effects.

Appendix F Data and institutional setting for the workplace application

This appendix collects additional information on the Veneto Worker History (VWH) sample used in Section 6: the full set of sample-selection restrictions (Appendix F.1), descriptive statistics on the resulting sample (Appendix F.2), firm-level mobility rates by firm size (Appendix F.3), robustness of the peer-group definition (Appendix F.4), and the institutional setting of the Italian labor market over the sample window (Appendix F.5).

Appendix Table 1: UW-Madison register data for spring semesters in 2019 and 2020 — full version including endogenous peer effects

	Spring 2019			Spring 2020		
	NLLS	CF	CF + endog.	NLLS	CF	CF + endog.
$\hat{\beta}$	0.249	0.169	0.026	0.049	−0.002	−0.076
	(0.030)	(0.033)	(0.042)	(0.026)	(0.033)	(0.036)
$\hat{\lambda}$			0.075			0.043
			(0.021)			(0.017)
$\frac{\hat{\beta}+\hat{\lambda}}{1-\hat{\lambda}}$			0.110			−0.034
			(0.034)			(0.033)
$\hat{\sigma}_{\bar{\alpha}_{(i)t}}$ plug-in	0.210			0.157		
$\hat{\sigma}_{\bar{\alpha}_{(i)t}}$ KSS		0.188	0.190		0.143	0.143
1-sd effect	0.052	0.032	0.021	0.008	−0.000	−0.005

Notes: Wild bootstrap standard errors for NLLS; The proposed standard errors based on $\hat{V}_n$ for cross-fit (CF). The *CF + endog.* column reports the joint estimator of (β, λ) in (23) with delta-method standard errors for the multiplier $(\hat{\beta} + \hat{\lambda})/(1 - \hat{\lambda})$.

F.1 Sample selection

We construct the analysis sample from the VWH database with the following restrictions, applied in order:

1. Restrict the panel to calendar years 1995–2001.
2. For workers with multiple employment relationships in the same year, retain only the primary job (highest annual earnings and weeks worked); ties are broken at random.
3. Restrict to workers aged 16–65.
4. Drop part-time contracts and apprentice contracts, because wages in those contracts are set under different rules than full-time regular employment.
5. Drop firms with more than 5,000 employees, following Cornelissen et al. (2017); these firms account for less than 2% of the observations.
6. Require each peer group (firm × broad occupation × year) to contain at least two workers. Workers in one-worker firms account for under 2% of observations and earn about 30% below the labor-market average.

7. Restrict to firms in the largest connected set of the worker-firm mobility graph (Abowd et al., 1999b); the largest connected set covers about 98% of the remaining observations.

8. Restrict further to the *leave-one-out* connected set: the set of firms that remains connected after the removal of any single mover. This is the condition required by the bias-corrected variance estimator of Kline et al. (2020b) and is the sample used in both Sections 6.2 and 6.3.

In observable characteristics — mean wages, age, tenure, gender composition, and occupational distribution — the leave-one-out sample is similar to the full sample, with a modest reduction in the number of firms and a slight increase in average firm size that reflects the stronger connectedness requirement.

F.2 Descriptive statistics

Table 2 reports descriptive statistics for the leave-one-out sample. The final sample contains approximately 4.8 million person-year observations covering 1.2 million workers and 69 thousand firms. The mean weekly wage is 833 euros (2003 prices). The average worker is 35 years old with about 5 years of tenure; 36% of workers change firms at least once over the seven-year window, and women make up 35% of the sample. Workers are predominantly in blue-collar occupations (68%), with white-collar workers at 30% and managers at 2%. Firms are small on average, consistent with the structure of the Italian labor market: the mean firm size is 21 employees with a median of 9; mean peer-group size is 10 with a median of 3.

F.3 Firm mobility by firm size

The mobility statistics cited in Section 6.2 as evidence for the strong-identification condition (Assumption 3(iv)) are documented at the firm level in Figure 2. On average, about 8% of workers move from another firm in a given year, and the average firm replaces around 20% of its workforce annually. As shown in the figure, this turnover rate is broadly similar across firm-size bins: even firms with more than 1,000 employees see annual turnover of around 15%. The pattern is informative for identification because the cross-fit estimator draws on movers and on coworker entry/exit around stayers, and the latter scales with firm-level turnover.

Appendix Table 2: Descriptive statistics for the leave-one-out sample, 1995–2001

	Mean	Standard Dev.	Median
Worker level			
Weekly wage (euros)	832.91	2731.75	701
Age	34.79	9.85	33
Tenure	4.60	5.02	3
Mover	0.36	0.48	
Woman	0.35	0.48	
Blue-collar	0.68	0.47	
White-collar	0.30	0.46	
Manager	0.02	0.14	
Firm level			
Firm size	21	92	9
Peer-group size	10	51	3
Annual mobility (workers)	5	36	2
Person-year observations		4,828,066	
Number of workers		1,203,965	
Number of firms		68,883	

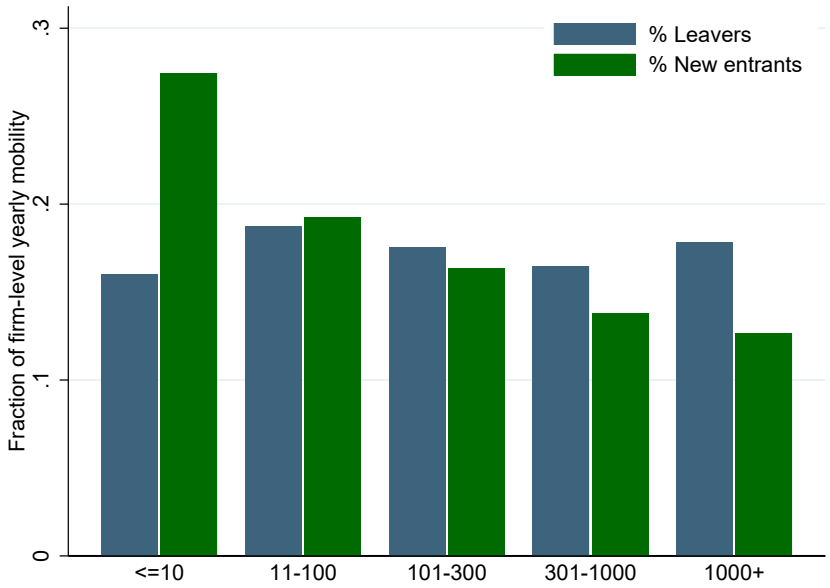
Notes: Sample is the leave-one-out connected set of VWH 1995–2001, after the sample restrictions enumerated in Appendix F.1. Wages are inflation-adjusted to 2003 prices.

F.4 Peer-group definition: robustness

The baseline peer-group definition is firm × broad occupation × year, with three broad occupational categories (blue-collar, white-collar, manager). Two concerns motivate robustness checks.

Managers as a separate group. Treating managers as a peer group distinct from blue- and white-collar workers may be conservative if managers in fact interact with the workers they supervise. As a robustness exercise, we reassign each manager to either blue- or white-collar based on the occupation she was promoted from. Where a worker has always been observed as a manager, we assign her to the modal non-manager occupation in the same firm-year (defaulting to blue-collar if all coworkers are blue-collar). Re-estimating equation (12) with this redefined peer group yields a peer coefficient nearly identical to the baseline, which is expected given that managers comprise only 2% of the sample.

Appendix Figure 2: Annual firm turnover by firm size



Notes: At the firm level, average fractions of newly entered and newly separated workers in a given year, plotted against firm size. VWH leave-one-out sample, 1995–2001.

Within-occupation heterogeneity. The broad-occupation definition can be too coarse if distinct occupations within a professional category (e.g., consultants and accountants within white-collar) do not in fact interact. Two observations bear on this concern. First, peer-group sizes are typically small (median 3), so cross-occupation interaction within a firm is plausible: in many firms, small white-collar teams span occupations and work jointly. Second, Portugal et al. (2024) document quantitatively similar peer-effect estimates under alternative occupation definitions using comparable Portuguese matched data. To the extent that residual measurement error in the peer-group definition remains, it generates the classical attenuation discussed in Cornelissen et al. (2017) and Nix (2016), and our reported coefficients can be read as a lower bound on the true peer effect.

F.5 Italian wage-setting institutions over the sample window

The sample window 1995–2001 sits within the relatively decentralized phase of Italian wage-setting institutions described in Guiso, Pistaferri and Schivardi (2005). A major restructuring of the Italian industrial-relations system in July 1993 reorganized

the relationship between sector-level and firm-level bargaining; we begin our sample in 1995 to allow the post-reform regime to settle and because firm-level information in the VWH is more accurate from 1995 onward. Within international comparisons, Guiso et al. (2005) report that the Iversen (1998) index of wage-bargaining centralization places Italy at the lower-centralization end of OECD countries over this period, with a value near 0.18 — closer to the United Kingdom (0.18), France (0.12), and Switzerland (0.25) than to the highly centralized Nordic economies (above 0.50). This decentralized institutional setting is the labor market in which we identify workplace peer effects, and our estimates should be read against that backdrop rather than against settings where wages are predominantly set at the national or sectoral level.